Analysis of a Generalized Retrial System with Coupled Orbits

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Introduction

- Consider retrial system with **coupled orbits**:
  retrial (transmission) rate depends on binary state **"busy-idle"** of other orbits.
- **Motivation**: In cognitive radio, a wireless node can know state of environment (other sources) and dynamically changes retrial rates to achieve **full spectrum utilization** [2, 3, 4].
- Current trend to dense networks increases the impact of **wireless interference** [4].

Model description

- Single server queuing system
- Class-i customers follow Poisson input (with parameter λ_i) and service time $S^{(i)}$ with **general distribution** and parameter

$$\gamma_i = \frac{1}{\mathsf{E}S^{(i)}} \in (0,\, \infty),\ i = 1, \ldots, N.$$

- Class-i customer joins the ith orbit if server is busy.
- Exponential class-i retrial time with parameter $\mu^i_{(\cdot)}$ depending on current states of other orbits: **busy or empty**.

Model description

- In more detail, consider $(N-1)$- dimensional vector

$$J(i) = \{j_1, \dots, j_{i-1}, j_{i+1}, \dots, j_N\},$$

the component j_i **is omitted**.

- Each component $j_k \in \{0, 1\}$: if $j_k = 1$, then orbit k is busy, otherwise, if $j_k = 0$, then orbit k is empty.
- Vector $J(i)$ describes the states of **other orbits** $k \neq i$.

Model description

- We call $J(i)$ a *configuration*.
- Define set $\mathcal{G}(i) = \{J(i)\} = \{$ all possible configurations $J(i)\}$.
- With configuration $J(i)$, orbit i has **given constant rate** $\mu_{J(i)}$.
- Denote set $M_i = \{\mu_{J(i)}\}$ - all possible rates when orbit i is busy.
- This model becomes **classic multiclass retrial if, for given** i, $\mu_{J(i)} = \mu_i$ **is a constant for all** $J(i)$.

Particular case: 3 orbits

- For 3 orbits: 1,2,3, capacity of each $\mathcal{M}_i$ and $\mathcal{G}(i)$ equals 4.
- Indeed, given orbit i, for 2 other orbits $j < k$, the following 4 configurations $J(i)$ are possible:

$$\begin{aligned}\mathcal{G}(i) &= \{J(i)\} = \Big\{(i_j = 0,\ i_k = 0), (i_j = 1,\ i_k = 0),\\ &\quad (i_j = 0,\ i_k = 1), (i_j = 1,\ i_k = 1)\Big\}. \end{aligned} \tag{1}$$

- In notation $\mu^i_{(\cdot)}$ the state of configuration $J(i)$ is reflected. For instance, μ^1_{01} = rate of configuration $J(1) = (i_2 = 0,\ i_3 = 1)$, and μ^3_{10} = rate of configuration $J(3) = (i_1 = 1,\ i_2 = 0)$.

Notation

$I(t)$ = idle time of server in $[0,\ t]$.

$B(t)$ = busy time of server in $[0,\ t]$, that is $I(t) + B(t) = t$.

Stationary idle and busy probability of server are defined:

$$\lim_{t\to\infty} \frac{I(t)}{t} = \mathsf{P}_0 = 1 - \mathsf{P}_b = 1 - \lim_{t\to\infty} \frac{B(t)}{t}. \tag{2}$$

$$\text{load coefficients: } \rho_i = \lambda_i/\gamma_i,\ \rho = \sum_{i=1}^{N} \rho_i.$$

$$\textbf{maximal rate} \text{ of orbit } i:\ \hat{\mu}_i = \max_{J(i)\in\mathcal{G}(i)} \mu_{J(i)};$$

$$\textbf{minimal rate} \text{ of orbit } i:\ \mu_i^0 = \min_{J(i)\in\mathcal{G}(i)} \mu_{J(i)}.$$

$B_i(t)$ = busy time server is occupied by class-i customers, in $[0,\ t]$.

Theoretical statements

The proofs use balance equations and regenerative method [1, 4].

Theorem 1. The stationary probability the **server is occupied by class-i customer** is

$$\mathsf{P}_b^{(i)} = \lim_{t\to\infty} \frac{B_i(t)}{t} = \rho_i, \ \ i = 1, \dots, N. \tag{3}$$

Let $\mathsf{P}_0^{(i)}$ be stationary probability server is **idle and orbit i is busy.**

Theorem 2. The following **bounds** hold:

$$\frac{\lambda_i}{\hat{\mu}_i}\rho \leqslant \mathsf{P}_0^{(i)} \leqslant \frac{\lambda_i}{\mu_i^0}\rho, \ \ i = 1, \dots, N. \tag{4}$$

Theoretical statements

For **classic retrial model** with $\mu_{J(i)} \equiv \mu_i$ it holds:

Theorem 3. Stationary probability that **server and orbit** i **are idle**:

$$P_{0,0}^{(i)} = 1 - \rho \left(1 + \frac{\lambda_i}{\mu_i}\right), \; i = 1, \dots, N. \tag{5}$$

Corollary. Inequality (4) becomes

$$P_0^{(i)} = \frac{\lambda_i}{\mu_i}\rho, \; i = 1, \dots, N. \tag{6}$$

Stability conditions

Theorem 4. Necessary stability condition:

$$\mathsf{P}_b = \sum_{i=1}^{N} \rho_i = \rho \leqslant \min_{1 \leqslant i \leqslant N} \left[\frac{\hat{\mu}_i}{\lambda_i + \hat{\mu}_i} \right] < 1. \tag{7}$$

Theorem 5. Sufficient stability condition:

$$\rho \leqslant \min_{1 \leqslant i \leqslant N} \left[\frac{\mu_i^0}{\lambda_i + \mu_i^0} \right] < 1. \tag{8}$$

Introduce the **metrics**

$$\Gamma_1 := \min_i \frac{\hat{\mu}_i}{\lambda_i + \hat{\mu}_i} - \rho, \ \Gamma_2 := \min_i \frac{\mu_i^0}{\lambda + \mu_i^0} - \rho. \tag{9}$$

$\Gamma_i > 0$ in **stability zone**, and $\Gamma_i \leqslant 0$, otherwise.

Symmetric model

Now we discuss an important special class: **symmetric systems.**

The classic retrial system with constant rate is symmetric, if all corresponding parameters are equal:

$$\lambda_i \equiv constant, \, \gamma_i \equiv \gamma, \; \mu_i \equiv \mu.$$

For system with **coupled orbits** the notion symmetry is **more flexible.** Now we illustrate it.

Symmetric model

We illustrate symmetry for model with 3 orbits with rates

$$M_i = \{\mu^i_{00},\ \mu^i_{10},\ \mu^i_{01},\ \mu^i_{11}\},\ i = 1, 2, 3,$$

corresponding to configurations

$$\mathcal{G}(i) = \{J(i)\} = (\{0,0\}, \{1,0\}, \{0,1\}, \{1,1\}).$$

In symmetric model,

$$M_1 = M_2 = M_3$$

but rates within M_i may be **different**.

Simulation setting

We consider: 3 orbits; $N(t)$ = orbit size.

(t-axis counts **"discrete events"**: arrivals, attempts, departures.)

Exponential $\exp(\gamma_i)$ **or Pareto service time distribution**:

$$F_i(x) = 1 - (\frac{x_0^i}{x})^\alpha, \; x \geqslant x_0^i \;\; (F_i(x) = 0, \; x \leqslant x_0^i),$$

with expectation

$$\frac{1}{\gamma_i} = \mathsf{E}S^{(i)} = \frac{\alpha\, x_0^i}{\alpha - 1}, \;\; \alpha > 1, \; x_0^i > 0, \; i = 1, 2, 3. \qquad (10)$$

Simulation: symmetric model

For the *symmetric exponential and Pareto models* we choose the following parameters:

$$\lambda_1 = \lambda_2 = \lambda_3 = 3;$$
$$M_1 = \{\mu^1_{00} = 20,\ \mu^1_{10} = 30,\ \mu^1_{01} = 15,\ \mu^1_{11} = 25\},$$
$$M_2 = \{\mu^2_{00} = 20,\ \mu^2_{10} = 30,\ \mu^2_{01} = 15,\ \mu^2_{11} = 25\},$$
$$M_3 = \{\mu^3_{00} = 20,\ \mu^3_{10} = 30,\ \mu^3_{01} = 15,\ \mu^3_{11} = 25\}. \qquad (11)$$

Simulation: exponential symmetric model

$\gamma_1 = \gamma_2 = \gamma_3 = 12$, implying $\Gamma_1 = 0.14, \Gamma_2 = -0.16$;

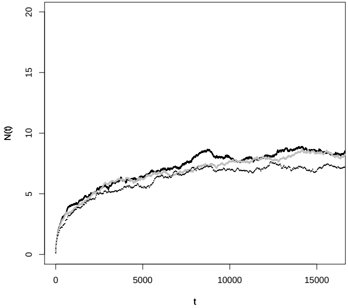

Figure: 1. Symmetric system, exponential service time. Condition (7) holds, condition (8) is violated: $\Gamma_1 > 0$, $\Gamma_2 < 0$; **but all orbits are stable.**

Simulation: exponential symmetric model

$\gamma_1 = \gamma_2 = \gamma_3 = 10$, implying $\Gamma_1 = 0, \Gamma_2 = -0.3$;

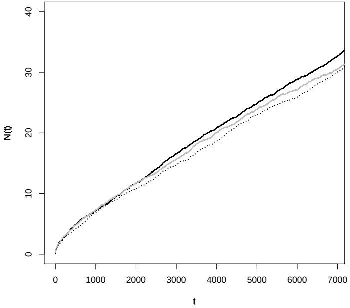

Figure: 3. Symmetric system, exponential service time. Conditions (7) and (8) are violated: $\Gamma_1 < 0$, $\Gamma_2 < 0$; all orbits are unstable.

Simulation: symmetric Pareto model

For Pareto service time, we select $\alpha = 2$ and shape parameter

$$x_0^i = \frac{1}{30} \text{ for orbit } i = 1, 2, 3. \tag{12}$$

This gives service rates $\gamma_1 = \gamma_2 = \gamma_3 = 15$.

All other parameters remain as in (11).

Simulation: symmetric Pareto model

$\Gamma_1 = 0.3$, $\Gamma_2 = 0.23$.

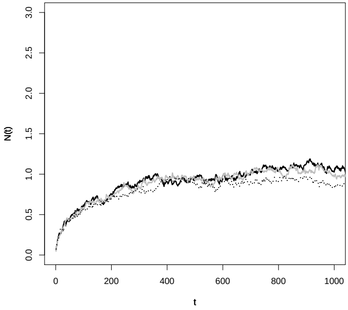

Figure: 4. Pareto service time. Both conditions (7), (8) hold: $\Gamma_1 > 0$, $\Gamma_2 > 0$; all orbits are stable.

Simulation: symmetric Pareto model

$x_0^i = \frac{1}{12}, \quad \gamma_i = 6, \quad \Gamma_1 = -0.6, \quad \Gamma_2 = -0.67;$

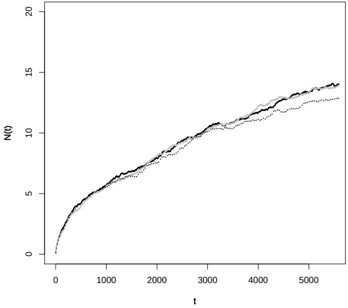

Figure: 5. Pareto service time. Conditions (7) and (8) are violated: $\Gamma_1 < 0$, $\Gamma_2 < 0$; all orbits are unstable.

Simulation: estimation of probability $P_0^{(i)}$

For the next experiments for the exponential model we choose the following parameters:

$$\lambda_1 = \lambda_2 = \lambda_3 = 3;$$
$$\gamma_1 = 10, \gamma_2 = 20, \gamma_3 = 30;$$
$$M_1 = \{\mu_{00}^1 = 20,\ \mu_{10}^1 = 30,\ \mu_{01}^1 = 15,\ \mu_{11}^1 = 20\},$$
$$M_2 = \{\mu_{00}^2 = 10,\ \mu_{10}^2 = 12,\ \mu_{01}^2 = 23,\ \mu_{11}^2 = 30\},$$
$$M_3 = \{\mu_{00}^3 = 25,\ \mu_{10}^3 = 30,\ \mu_{01}^3 = 7,\ \mu_{11}^3 = 20\}. \qquad (13)$$

$\Gamma_1 = 0.35, \Gamma_2 = 0.28$; **Both stability conditions are satisfied.**

Simulation: estimation of probability $P_0^{(1)}$

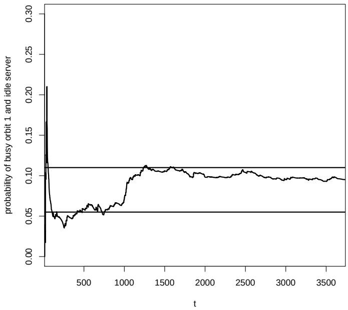

Figure: 6. Exponential service time. Conditions (7) and (8) hold. Estimate of probability $P_0^{(1)} = P(\text{busy orbit 1, idle server})$ satisfies bounds (4).

Simulation: estimation of probability $P_0^{(2)}$

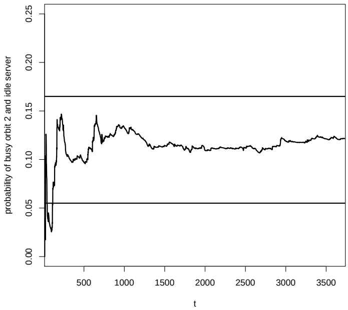

Figure: 7. Exponential service time. Conditions (7) and (8) hold. Estimate of probability $P_0^{(2)} = P(\text{busy orbit 2, idle server})$ satisfies bounds (4).

Simulation: estimation of probability $P_0^{(3)}$

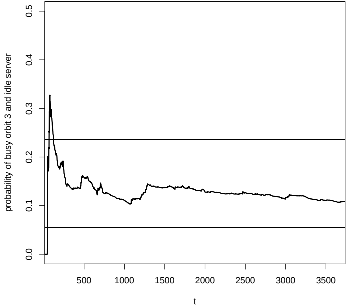

Figure: 8. Exponential service time. Conditions (7) and (8) hold. Estimate of probability $P_0^{(3)} = P(\text{busy orbit 3, idle server})$ satisfies bounds (4).

Conclusion

To verify some theoretical results, we simulate a 3 -class retrial system with independent Poisson inputs and the coupled orbits:

- a class-i customer meeting server busy joins virtual infinite capacity i-orbit;
- the orbit i retrial rate depends on configuration of other orbits: *busy or idle*.
- Simulation indicates that the **necessary stability condition is indeed stability criterion**.

References

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Thanks for your attention!

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