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Generating Systems of Non-negative Linear Diophantine Equations

homANLDE system

Associated with CF-grammar homogeneous system of non-negative linear diophantine equations

$$\mathrm{E}^m(I_0, I_1, \ldots, I_n)x = Ax, \quad \mathrm{E}^m(I_0, I_1, \ldots, I_n) \in \{0, 1\}^{n \times m}, \quad A \in \mathbb{Z}_+^{n \times m}.$$

Partition of finite natural scale interval $I^n = \{I_0, I_1, \ldots, I_n\}$, $I_k \subseteq \{1, 2, \cdots\}$.

Partition matrix $\mathrm{E}^m(I^n) = \mathrm{E}^m(I_0, I_1, \ldots, I_n) \in \{0, 1\}^{n \times m}$:

$$\mathrm{E}^m_{k,i}(I^n) = 1 \overset{\text{def}}{\Longleftrightarrow} i \in I_k, \quad k = 1, 2, \ldots, n, \; i = 1, 2, \ldots, m.$$

Hilbert basis — a set of undecomposable solutions $\mathcal{H} = \{h^{(1)}, \ldots h^{(q)}\}$.

Generating problem

Build homANLDE system and its Hilbert basis based on given parameters.

<u>Input</u>: generating parameters — number of equations m and unknowns m, maximum size of coefficients, maximum size of Hilbert basis, etc.

<u>Output</u>: $\mathrm{E}^m(I_0, I_1, \ldots, I_n), A, \mathcal{H}$.

Generating algorithms used in the Web-SynDic system, designed for remote demonstration and testing of solving algorithms for homANLDE systems.

http://websyndic.cs.karelia.ru

Transformation of arbitrary homANLDE system

Theorem 1 *Let an arbitrary homANLDE system with n equations and m unknowns:*

$$\mathrm{E}^m(I_1, \ldots, I_n)x = Ax. \tag{1}$$

The problem of searching Hilbert basis for system (1) can be reduced to the problem of searching Hilbert basis either for the equation:

$$\sum_{i \in I_k^*} x_i = \sum_{i \in J} a_{ki} x_i, \tag{2}$$

for some $k \in \mathbb{N}_n$, where $I_k^* \subseteq I_k$ and $J \subseteq \mathbb{N}_m$; or for the system of equations:

$$\sum_{i \in I_{k_l}^*} x_i = \sum_{j \in J_{k_l}} x_j, \quad = \overline{1, p} \tag{3}$$

for some $p \in \mathbb{N}_n$, where $I_{k_l}^* \subseteq I_{k_l}$ and $\bigcup\limits_{l=1}^{p} J_{k_l} = \bigcup\limits_{j=1}^{p} I_{k_l}^* \subseteq \bigcup\limits_{k=1}^{n} I_k = \mathbb{N}_m.$

homANLDE system generating algorithms

Jordan algorithm. homANLDE systems like (3) with unitary matrix and addition an unknowns with zero values. Common system view $(\mathbb{I}|B)\,x = (\mathbb{I}|B + \Delta)$. Published in 2005 year.

Gauss algorithm. homANLDE equations like (2). In each step of reverse transformation one unknown is added. Common system view $(\mathbb{I}|B)\,x = (\mathbb{O}|A_+|B + \Delta)$. Published in 2005 year.

Extgauss algorithm. homANLDE equations like (2). In each step of reverse transformation one or more unknowns are added.

Symsystem algorithm. homANLDE systems like (3) in common case.

Comsystem algorithm. homANLDE equations like (2) or homANLDE systems like (3). In case of build (3) the system adds by unknowns with zero values. In each step of reverse transformation one or more unknowns are added.

Extgauss algorithm

The idea is to build equation (2) in common view:

$$\sum_{i=k_{n-1}}^{k_n-1} x_i = \sum_{i=k_n}^{m} a_{ni} x_i, \, n \leq k_{n-1} \leq k_n - 1 < m, \, a_{ni} > 0$$

and execute reverse transformation.

The homANLDE system common view:

$$\left(\, \mathbb{I}' \, \middle| \, B \, \right) x = \left(\, \mathbb{O} \, \middle| \, A_+ \, \middle| \, B + \Delta \, \right) x$$

Formal description:

Input: n — number of equation, m — number of unknowns, $n < m$.

Output: $(\mathrm{E}^m, A, \mathcal{H})$.

1. generating randomly a number $p \in \{0, \ldots, m - n - 1\}$ and sequence $0 < k_1 < \cdots < k_n = n + p + 1$;

2. generating randomly a matrix $B \in \{0, 1\}^{n \times (m-n-p)}$, $\forall k \, \sum_{i=1}^{n} B_{ik} = 1$;

3. generating randomly a matrix $\Delta \in \mathbb{Z}_+^{n \times (m-n-p)}$, $\forall j \, \Delta_{nj} > 0$;

4. generating randomly a matrix A_+, $A_+ = \begin{pmatrix} a_{1k_1} & \cdots\cdots\cdots\cdots & a_{1(n+p)} \\ 0 & \cdots & 0 \, a_{2k_2} & \cdots & a_{2(n+p)} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ 0 & \cdots\cdots\cdots\cdots & 0 \end{pmatrix}$, $a_{ij} \geqslant 0$,

$a_{it} > 0$, $k_i \leqslant t < k_{i+1}$;

5. Compute matrix $\mathbb{I}'$, $\mathbb{I}' = \begin{pmatrix} 1 & \cdots & 1 \, 0 & \cdots\cdots\cdots\cdots & 0 \\ 0 & \cdots & 0 \, 1 & \cdots & 1 \, 0 & \cdots\cdots\cdots & 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ 0 & \cdots\cdots\cdots\cdots & 0 \, 1 & \cdots & 1 \end{pmatrix}$;

6. $\mathrm{E}^m(I^n) = \left(\, \mathbb{I}' \, \middle| \, B \, \right)$, $A = \left(\, \mathbb{O} \, \middle| \, A_+ \, \middle| \, B + \Delta \, \right)$;

7. Using reverse transformation build Hilbert basis $\mathcal{H}$.

Symsystem algorithm

The idea is to generate homANLDE system (3). The matrix of coefficients $E-A$ corresponds to the incidence matrix of some directed graph. The basis solution corresponds to the circuit.

The homANLDE system common view

$$\mathrm{E}^m(I_1,\ldots,I_n)x = \mathrm{E}^m(J_1,\ldots,J_n)x$$

Formal description:

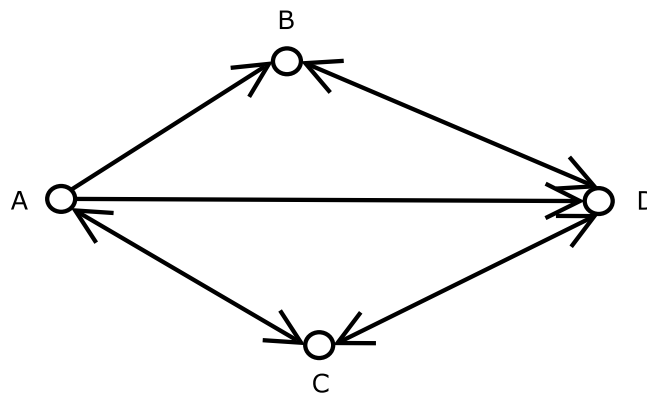**Input:** n — number of equations, m — number of unknowns, $0 < n < m$.

Output: $(\mathrm{E}^m(I^n), \mathrm{E}^m(J^n), \mathcal{H})$.

1. Generating randomly matrix $\mathrm{E}^m(I^n)$ and $\mathrm{E}^m(I^n)$;

2. Compute Hilbert basis $\mathcal{H}$.

homANLDE system example

$$x_{AB} + x_{AC} + x_{AD} = x_{CA}$$
$$x_{BA} + x_{BD} = x_{AB} + x_{DB}$$
$$x_{CA} + x_{CD} = x_{AC} + x_{DC}$$
$$x_{DA} + x_{DB} + x_{DC} = x_{AD} + x_{BD} + x_{CD}$$

Corresponded directed graph

Set of circuits
$$ABDCA, \quad ADCA$$

Hilbert basis
$$\forall i \ \ x_i = 0, \quad x_{AB} = 1, \ x_{BD} = 1, \ x_{DC} = 1, \ x_{CA} = 1$$
$$\forall i \ \ x_i = 0, \quad x_{AD} = 1, \ x_{DC} = 1, \ x_{CA} = 1$$

Comsystem algorithm

The idea is to generate either equation (2) or homANLDE system (3) and process reverse transformation.

The homANLDE system view:

$$\left(\begin{array}{c|c|c} E' & B \\ \hline \mathbb{O} & B'' & B' \end{array}\right) x = \left(\begin{array}{c|c|c} A_+ & B + \Delta \\ \hline \mathbb{O} & B'' + \Delta' & B''' \end{array}\right) x$$

Formal description:

Input: n — number of equations, m — number of unknowns, $n \leqslant m$.

Output: $(\mathrm{E}^m, A, \mathcal{H})$.

1. Generate randomly numbers $0 \leqslant q \leqslant n$, $0 \leqslant p < m - q$, $0 \leqslant t < m - q - p$ and sequence $0 < k_1 < \cdots < k_q = q + p + 1$;

2. Generate randomly matrix $B \in \{0,1\}_+^{q \times (m-q-p)}$, $B' \in \{0,1\}_+^{(n-q) \times (m-q-p-t)}$, $B'' \in \{0,1\}_+^{(n-q) \times t}$, $\forall j = 1, \ldots, m - q - p - t$ $\sum_{i=1}^{q} B_{i(t+j)} + \sum_{i=1}^{n-q} B'_{ij} = 1$, $\forall j = 1, \ldots, t$ $\sum_{i=1}^{q} B_{ij} + \sum_{i=1}^{n-q} B''_{ij} = 1$;

3. Generate randomly matrix $B''' \in \{0,1\}^{(n-q) \times (m-q-p-t)}$, $\forall j$ $\sum_i B'''_{ij} = 1$;

4. Generate randomly matrix $\Delta \in \mathbb{Z}_+^{n \times (m-q-p)}$, $\forall j$ $\Delta_{qj} > 0$;

5. Generate randomly matrix $\Delta' \in \mathbb{Z}_+^{(n-q) \times t}$, $\forall j$ $\sum_i \Delta'_{ij} > 0$;

6. Generate randomly matrix $A_+ \in \mathbb{Z}^{q \times (q+p)}$, $A' = $
$$\begin{pmatrix} 0 & \ldots & 0 & a_{1k_1} & \ldots & a_{1(k_2-1)} & a_{1k_2} & \ldots\ldots\ldots & a_{1(q+p)} \\ 0 & \ldots & 0 & 0 & \ldots & 0 & a_{2k_2} & \ldots\ldots\ldots & a_{2(q+p)} \\ \multicolumn{9}{c}{\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots} \\ 0 & \multicolumn{5}{c}{\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots} & 0 & \ldots & 0 \end{pmatrix};$$

7. Generate randomly matrix $E' \in \mathbb{Z}^{q \times (q+p)}$, $E' = $
$$\begin{pmatrix} 1 & \ldots & 1 & 0 & \multicolumn{4}{c}{\ldots\ldots\ldots\ldots\ldots} & 0 \\ 0 & \ldots & 0 & 1 & \ldots & 1 & 0 & \ldots\ldots\ldots & 0 \\ \multicolumn{9}{c}{\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots} \\ 0 & \multicolumn{5}{c}{\ldots\ldots\ldots\ldots\ldots\ldots\ldots} & 0 & 1 & \ldots & 1 \end{pmatrix};$$

8. $\mathrm{E}^m(I^n) = \left(\begin{array}{c|c|c} E' & B \\ \hline \mathbb{O} & B'' & B' \end{array}\right)$, $A = \left(\begin{array}{c|c|c} A_+ & B + \Delta \\ \hline \mathbb{O} & B'' + \Delta' & B''' \end{array}\right)$;

9. Using reverse transformation compute Hilbert basis $\mathcal{H}$.

An example of transformation homANLDE system to (2)

homANLDE system

$$\begin{cases} x_1 + x_2 &= 2x_2 + x_3 + 3x_4 \\ x_3 &= 2x_4 + x_5 \\ x_4 + x_5 &= 3x_2 + 2x_5 \end{cases}$$

The sum of equations

$$-x_1 + 4x_2 + 4x_4 + 2x_5 = 0$$

Exclude equation 1 and unknown x_1

$$x_1 = x_2 + x_3 + 3x_4 = T_1(x_2, x_3, x_4, x_5)$$

homANLDE system after transformation

$$\begin{cases} x_3 &= 2x_4 + x_5 \\ x_4 + x_5 &= 3x_2 + 2x_5 \end{cases}$$

The sum of equations

$$3x_2 - x_3 + x_4 + 2x_5 = 0$$

Exclude equation 1 and unknown x_3

$$x_3 = 2x_4 + x_5 = T_2(x_2, x_4, x_5).$$

homANLDE system transforms to the equation (2)

$$x_4 + x_5 = 3x_2 + 2x_5$$

Hilbert basis for the equation

$$\mathcal{H}^2 = \left\{ \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\} \begin{matrix} : x_2 \\ : x_4 \\ : x_5 \end{matrix}$$

Using rules for x_1 and x_3 compute Hilbert basis

$$\mathcal{H} = \left\{ \begin{pmatrix} 8 \\ 3 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 6 \\ 0 \\ 3 \\ 1 \\ 1 \end{pmatrix} \right\}$$

An example of transformation homANLDE system to (3)

homANLDE system

$$\begin{cases} x_1 + x_3 &= x_1 + 3x_3 \\ x_2 &= x_2 + 5x_3 \\ x_4 &= 2x_1 + 3x_2 + 7x_3 \end{cases}$$

The sum of equations

$$2x_1 + 3x_2 + 14x_3 - x_4 = 0$$

Exclude equation 3 and unknown x_4

$$x_4 = 2x_1 + 3x_2 + 7x_3 = T_1(x_1, x_2, x_3)$$

homANLDE system after transformation

$$\begin{cases} x_1 + x_3 &= x_1 + 3x_3 \\ x_2 &= x_2 + 5x_3 \end{cases}$$

The sum of equations

$$7x_3 = 0$$

Set $x_3 = 0$ and exclude x_3. homANLDE system transforms to (3)

$$\begin{cases} x_1 &= x_1 \\ x_2 &= x_2 \end{cases}$$

Hilbert basis

$$\mathcal{H}^1 = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} \begin{matrix} : x_1 \\ : x_2 \end{matrix}$$

Using rule for x_4 compute Hilbert basis for homANLDE system:

$$\mathcal{H} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 3 \end{pmatrix} \right\}$$