Some analytical aspects of regenerative simulation of fluid models

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Abstract

We discuss the estimation of the loss probability in a queueing system with finite buffer. We apply regenerative technique combined with the so-called Delta-method to construct confidence interval for the stationary loss probability. This work is supported by Russian Foundation for Basic research, project No 10-07-00017 and done in the framework of the Strategy development Program for 2012-2016 "PetrSU complex in the research-educational space of European North: strategy of the innovation development.

1 Introduction

We consider a single server queue with finite buffer of size b, constant service rate C and input process $A(t) = mt + N(0, t\sigma^2)$, consisting of deterministic linear process mt with positive drift $m > 0$, and Brownian motion $N(0, t\sigma^2)$. The workload process in this system is described by the well-known (discrete time) Lindley recursion:

$$Q_n = \min((Q_{n-1} - C + X_n)^+, b),\ n = 1, 2, \dots, \tag{1.1}$$

where

$$X_n := A(n+1) - A(n) =_{st} m + N(0, \sigma^2)$$

are the i.i.d increments of the input process at instants $n = 1, 2, \dots$. We denote this system as Bi/D/1/b system. A motivation of this model can be found in [3]. Denote by $L_b(T)$, the total lost workload in interval $[0, T]$, that is

$$L_b(T) := \sum_{k=1}^{T} (Q_{k-1} - C + X_k - b)^+,\ T = 1, 2, \dots$$

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The time average loss $\mathsf{P}_\ell(b,T)$ in this system during (discrete-time) interval $[0,T]$, is defined as the ratio of the amount of lost workload and the total arrived workload, during this interval, that is

$$\mathsf{P}_\ell(b,T) := \frac{L_b(T)}{A(T)}. \tag{1.2}$$

Because the buffer is finite, the system is stable and the loss ratio, as $T \to \infty$, converges to a stationary loss probability $\mathsf{P}_\ell(b)$, that is

$$\mathsf{P}_\ell(b) := \lim_{T\to\infty} \mathsf{P}_\ell(b,T) = \frac{\mathsf{E}(Q+X-C-b)^+}{m}, \tag{1.3}$$

where Q is the stationary workload and X is a generic element of X_n. The following heuristic expression given in [4]

$$\mathsf{P}_\ell(b) \approx \frac{\mathsf{P}_\ell(0)}{\mathsf{P}(Q>0)}\mathsf{P}(Q>b), \tag{1.4}$$

allows to calculate the loss probability provided there is an explicit formula (or a satisfactory approximation) for the overflow probability $\mathsf{P}(Q>x)$ in the associated infinite buffer system. In our case, it is possible to use the following continious-time approximation (see [5]):

$$\mathsf{P}(Q>x) \approx \exp\left(-2\cdot\frac{C-m}{\sigma}\cdot x\right). \tag{1.5}$$

Moreover, it is easy to calculate $\mathsf{P}_\ell(0)$, namely,

$$\begin{aligned}\mathsf{P}_\ell(0) &= \frac{\mathsf{E}(X-C)^+}{m} \\ &= \frac{1}{m\sigma\sqrt{2\pi}}\int_c^\infty (x-c)e^{-(x-m)^2/2\sigma^2}dx. \end{aligned} \tag{1.6}$$

Thus results (1.4), (1.5), (1.6) allow to find an approximation of the overflow probability $\mathsf{P}_\ell(b)$ (in the following it will be denoted as P_ℓ) in our model.

2 Regenerative approach

In this section, we show how to estimate the steady-state loss probability P_ℓ using the regenerative approach. First we construct regeneration points for the content process. (More details can be found in [3].) Let $\beta_0 = 0$ and

$$\beta_{k+1} = \min\{n > \beta_k : Q_{n-1} > 0,\, Q_n = 0\},\ k \geq 1, \tag{2.1}$$

where, Q_n is defined in (1.1). Denote by L_i and A_i the workload lost and arrived per the ith regeneration cycle, respectively, with the corresponding generic elements L and A. It follows from the regenerative method, that

$$\mathsf{P}_\ell = \frac{\mathsf{E}L}{\mathsf{E}A}.$$

To apply the regenerative confidence estimation, we generate i.i.d. replications $L_1, .., L_n$, $A_1, ..., A_n$, to estimate the unknown means $\mathsf{E}L$, $\mathsf{E}A$ and the probability P_ℓ as

$$\widehat{L} := \frac{1}{n}\sum_{i=1}^{n} L_i,\ \widehat{A} := \frac{1}{n}\sum_{i=1}^{n} A_i,\ \widehat{\mathsf{P}}_\ell := \frac{\widehat{L}}{\widehat{A}}, \tag{2.2}$$

respectively. Using Delta-method, it is possible to show that

$$\sqrt{n}\left(\widehat{\mathsf{P}}_\ell - \mathsf{P}_\ell\right) \Rightarrow N(0,\eta^2),\ \ n \to \infty, \tag{2.3}$$

where $\Rightarrow$ stands for weak convergence and

$$\eta^2 = \frac{\mathsf{E}\left[L - A\cdot \mathsf{P}_\ell\right]^2}{(\mathsf{E}A)^2}.$$

(See [1, 2] for more detail on Delta-method.) In turn, to estimate η^2 we apply standard sample estimation

$$\widehat{\eta}^2 := \frac{\frac{1}{n-1}\sum\limits_{i=1}^{n}(L_i - \widehat{\mathsf{P}}_\ell A_i)^2}{\left(\frac{1}{n}\sum\limits_{i=1}^{n} A_i\right)^2} \tag{2.4}$$

Based on (2.3) one can obtain the following $(1-\gamma/2)\%$ asymptotical confidence interval for P_ℓ:

$$\left[\widehat{\mathsf{P}}_\ell - \frac{z_\gamma}{\sqrt{n}},\ \widehat{\mathsf{P}}_\ell + \frac{z_\gamma}{\sqrt{n}}\right], \tag{2.5}$$

where $z_\gamma = \widehat{\eta}\Phi^{-1}\left(\frac{\gamma}{2}\right)$, $\Phi^{-1}(x)$ is the inverse of Laplace function and γ is a given confidence probability.

3 Numerical examples

In this section, we present a few numerical results on confidence estimation of stationary loss probability in the above considered system $Bi/D/1/b$.

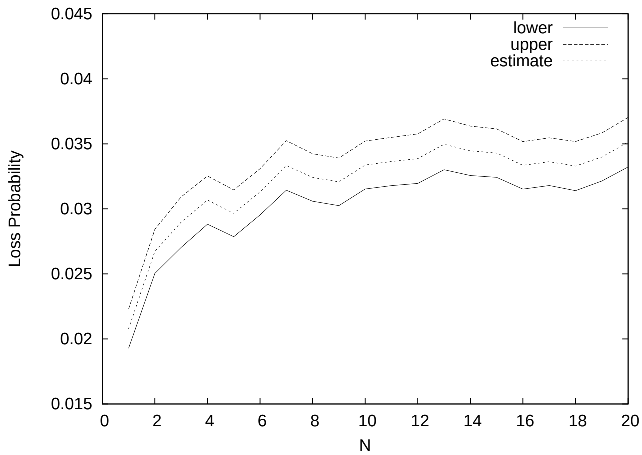

Figure 1: 95% Confidence interval for P_ℓ in Bi/D/1/4

First of all regeneration points are constructed as in (2.1). Then we calculate the confidence interval for the probability P_ℓ according to expression (2.5).

First we analyze a dependence the simulation results on the *step of digitization* $h := h_N = 1/N$ where $N = 1, 2, \ldots$ Figure 1 shows 95% confidence interval as a function of N for the model with parameters $m = 0.8$, service rate $C = 1$, buffer size $b = 4$ for a fixed simulation length of T time slots and $T = 10^5$. It is seen that confidence interval width is rather insensitive to the selection of the concrete value of step digitization h in a wide range of values of N. This remark may be useful to simplify simulation procedure and, in particular, to save simulation time.

Figure 2 compares the simulation results (based on the regenerative approach described above) with the approximation (1.4), where the following parameters are used: $C = 1$; $m = 0.7$; $T = 10^6$, and the confidence probability is $1 - \gamma = 0.95$.

Finally, we study the dependence of the simulation results on the choice of regeneration points. Namely, we form the *subsequences* of regeneration points $\{\beta_k^s\}$ of (2.1) for arbitrary (fixed) s as $\beta_i^s = \beta_{si}$ where $s = 1, 2, \ldots$ and $i = 0, 1, \ldots$ Figure 3 shows the 95% confidence interval width (for the loss probability P_ℓ) vs. the parameter s. The following parameters are used in simulation: $b = 4$; $C = 1$; $m = 0.8$; $T = 10^5$. Again, simulation demonstrates an insensitivity to the choice of the parameter s, and, in our opinion, it can be exploit to speed-up estimation.

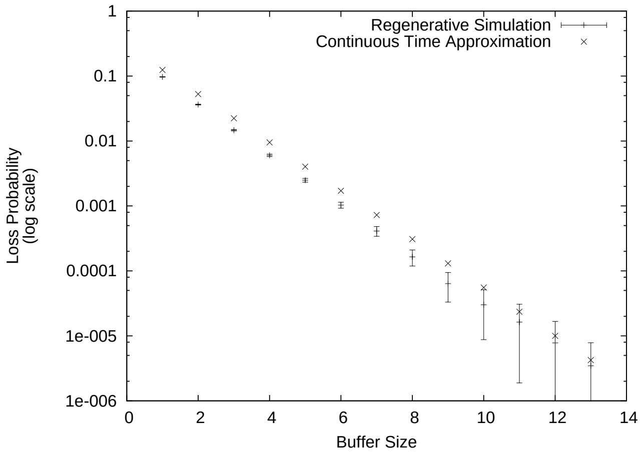

Figure 2: Estimate of P_ℓ in Bi/D/1/b: regenerative method vs. approximation (1.4)

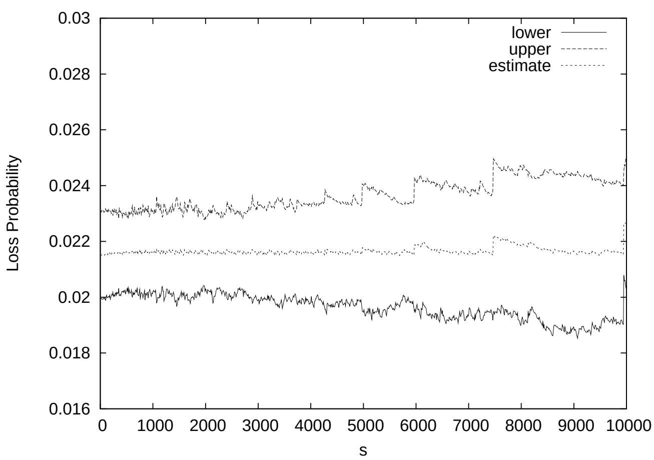

Figure 3: Dependence of the confidence interval width on the choice of subsequences of regeneration points

4 Conclusion

The estimation of the stationary loss probability in a single-server fluid queue with a Gaussian input using the regeneration approach is considered. A known approximation of the loss probability via the overflow probability is used to verify an accuracy the estimation based on the regenerative simulation. Some numerical results are presented.

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