Regenerative simulation of the overflow probability in the finite buffer queue with Brownian input

Ruslana Goricheva,

Oleg Lukashenko,

Dr. Evsey Morozov,

Dr. Michele Pagano

Gaussian models

Stationary input: $A(s,t) =_d \mathbb{N}(m \cdot (t-s), \upsilon(t-s))$

$\upsilon(t)$ - variance $\quad m$ - mean rate

$$A(t) := A(0,t)$$

Covariation function:

$$\Gamma(s,t) := Cov(A(s), A(t)) = \frac{1}{2}(\upsilon(t) + \upsilon(s) - \upsilon(t-s))$$

Fluid model

Input traffic:

$$A(t) = mt + \sqrt{m} \cdot X(t)$$

- m = mean input rate
- X(t) = random centred Gaussian process with covariance

$$\Gamma(t,s) = \mathbf{E}[X(t)X(s)]$$

Service rate:

Constant service rate C

Overflow probability:

$$P_l = P\left(\sup_{t \in S} \left(A(t) - Ct \right) \geq b \right)$$

Estimation methods:

- MC estimators
- Importance Sampling (IS)
- Regenerative method

Relative Error

$$RE(\hat{p}_l) \triangleq \frac{\sqrt{Var(\hat{p}_l)}}{\mathrm{E}[\hat{p}_l]}$$

For MC-estimator

$$RE(\hat{p}_l) \sim \frac{1}{\sqrt{p_l N}} \quad \textit{as} \quad p_l \to 0$$

For $p_l \to 0,$ the number N of samples must be sufficiently large

Functional Central Limit Theorem

$$\lim_{n\to\infty}\left\{\frac{\left(A(tn)-nmt\right)}{\sqrt{nm}},\ t\geq 0\right\} =_d \left\{B\ (t),\ t\geq 0\right\},$$

which means that

$$A(tn)\approx nmt+\sqrt{nm}B(t)$$

$\left\{B\ (t),\ t\geq 0\right\}$ - Brownian motion

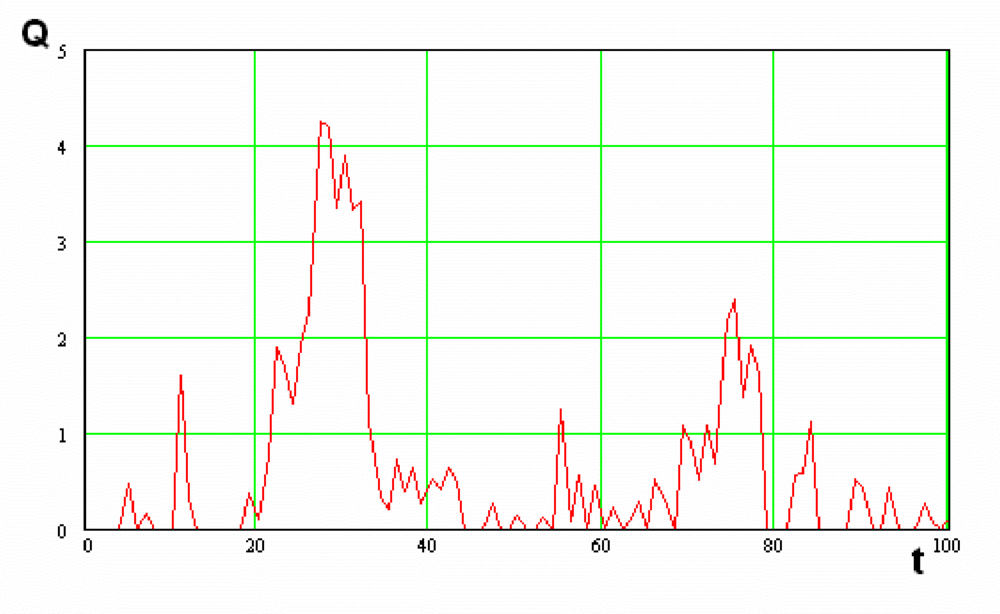
Queue with Brownian input

Input traffic:

$$A(t) = mt + \sqrt{m}\,B(t)$$

Queue length (in discrete scale):

$$Q(t) = \max\left(0, Q(t-1) - C + m + \sqrt{m}\left(B(t) - B(t-1)\right)\right),\ \ \mathrm{t} = 0, 1, \ldots$$



Regenerative simulation

$$\lim_{j\to\infty}\frac{1}{j}\sum_{i=0}^{j} f(X_i) = \frac{E\sum_{i=1}^{\beta} f(X_i)}{E\beta} = Ef(X) := r,$$

Confidence interval for r : $\left[r_k - \frac{z_\gamma s(k)}{\bar{\alpha}_k\sqrt{k}}, r_k + \frac{z_\gamma s(k)}{\bar{\alpha}_k\sqrt{k}}\right],$

- $r_k = \frac{\overline{Y_k}}{\overline{\alpha_k}},$
- $\alpha_i = \beta_i - \beta_{i-1}, \quad i > 0, \quad \overline{\alpha_k} = \frac{1}{k}\sum_{i=1}^{k}\alpha_i,$
- $Y_i = \sum_{i=\beta_{j-1}}^{\beta_j - 1} f(X_i), \quad j > 0, \quad \overline{Y_k} = \frac{1}{k}\sum_{i=1}^{k} Y_i,$
- $z_\gamma \equiv \Phi^{-1}\left(\frac{1-\gamma}{2}\right),$
- $\sigma^2 = E[(Y_1 - r\beta_1)^2], \; s^2(k) \to \sigma^2.$

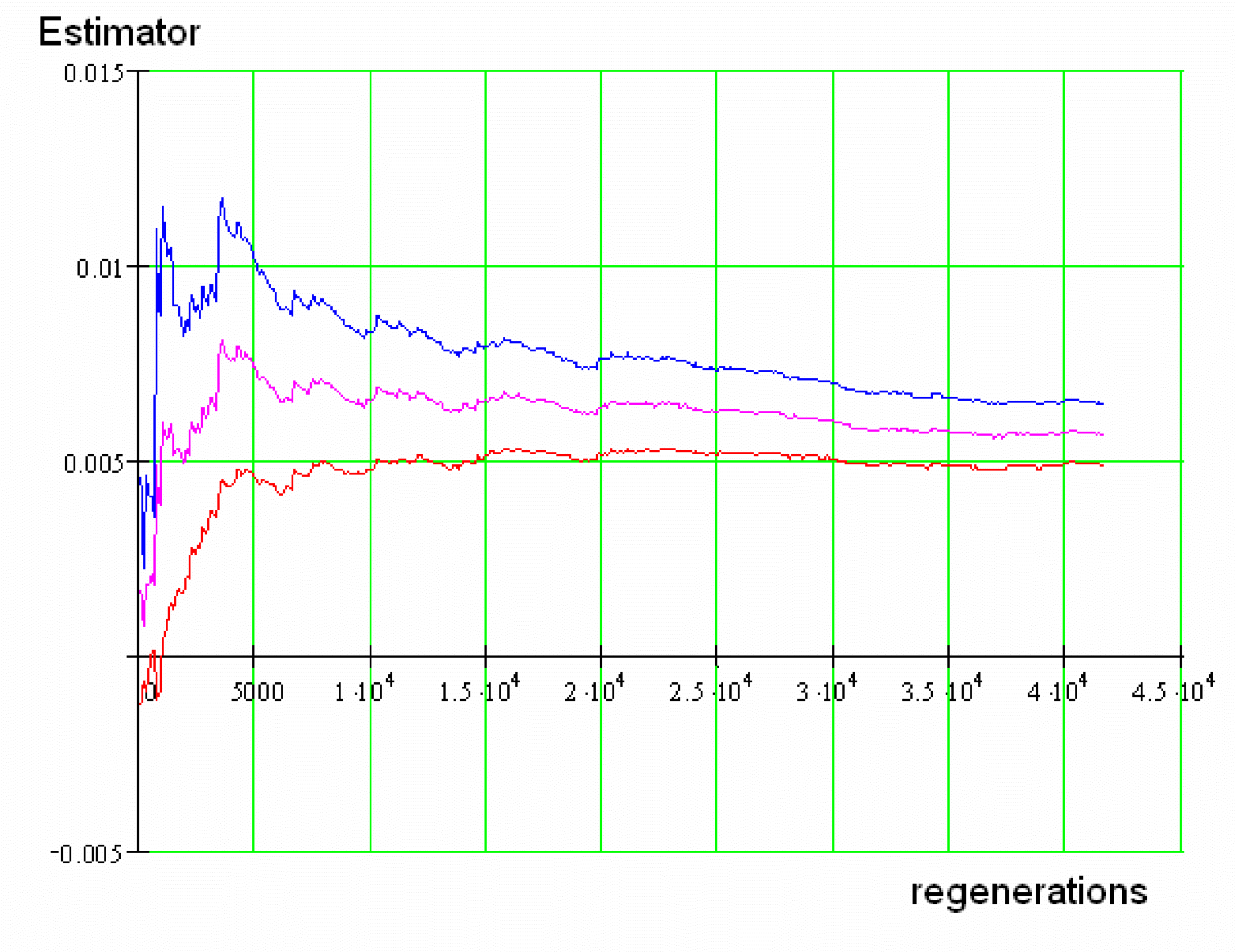
Estimation of loss probability

$$\beta_{k+1} = \min\{t > \beta_k : Q_t = 0\},\ \beta_0 = 0$$

- Q_t - workload of a system at time t
- EL - mean lost work per cycle
- EA - mean workload arrived per cycle
- $L_n(t)$ - lost work in [0; t], n – buffer size

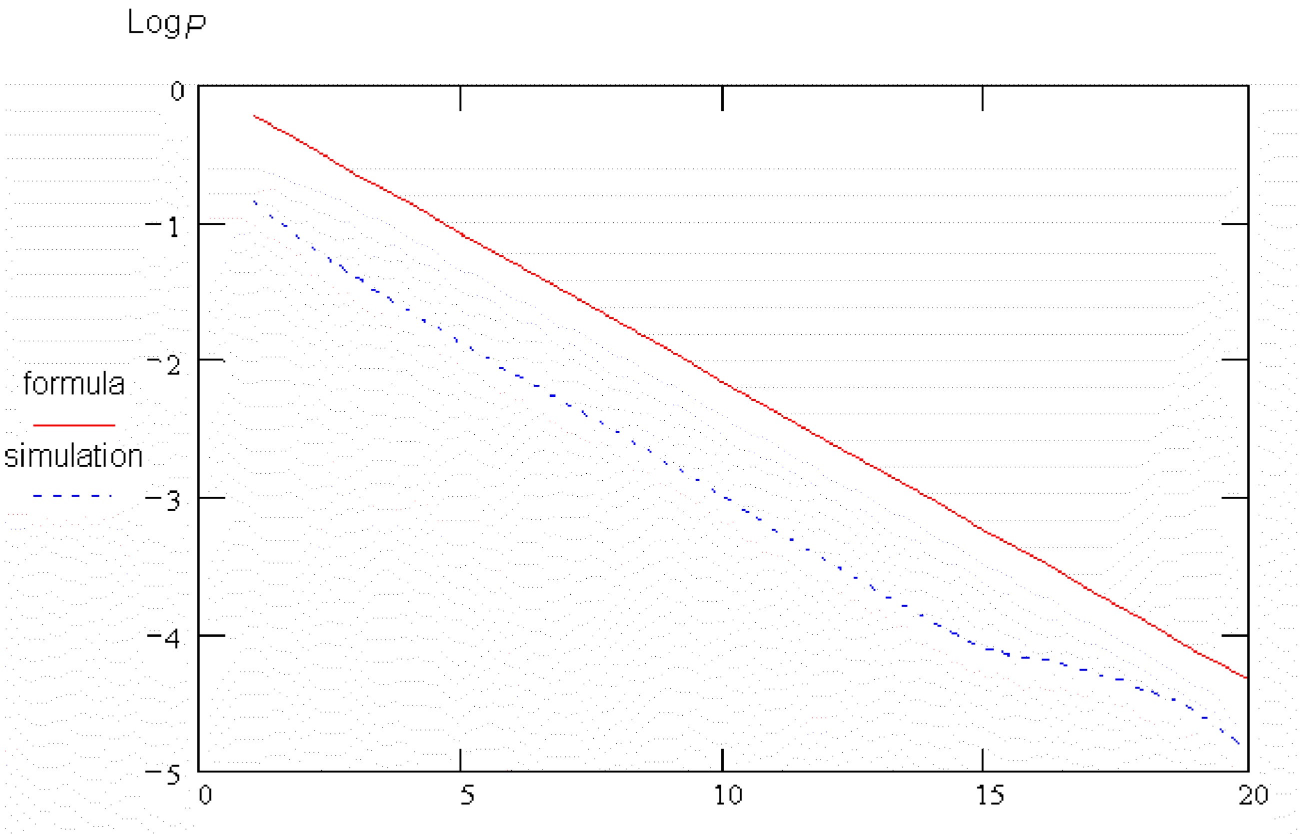
$$\lim_{t \to \infty} \frac{L_n(t)}{A(t)} = \frac{EL}{EA} \equiv P_l$$

Confidence interval for P_l in BM/D/1/n



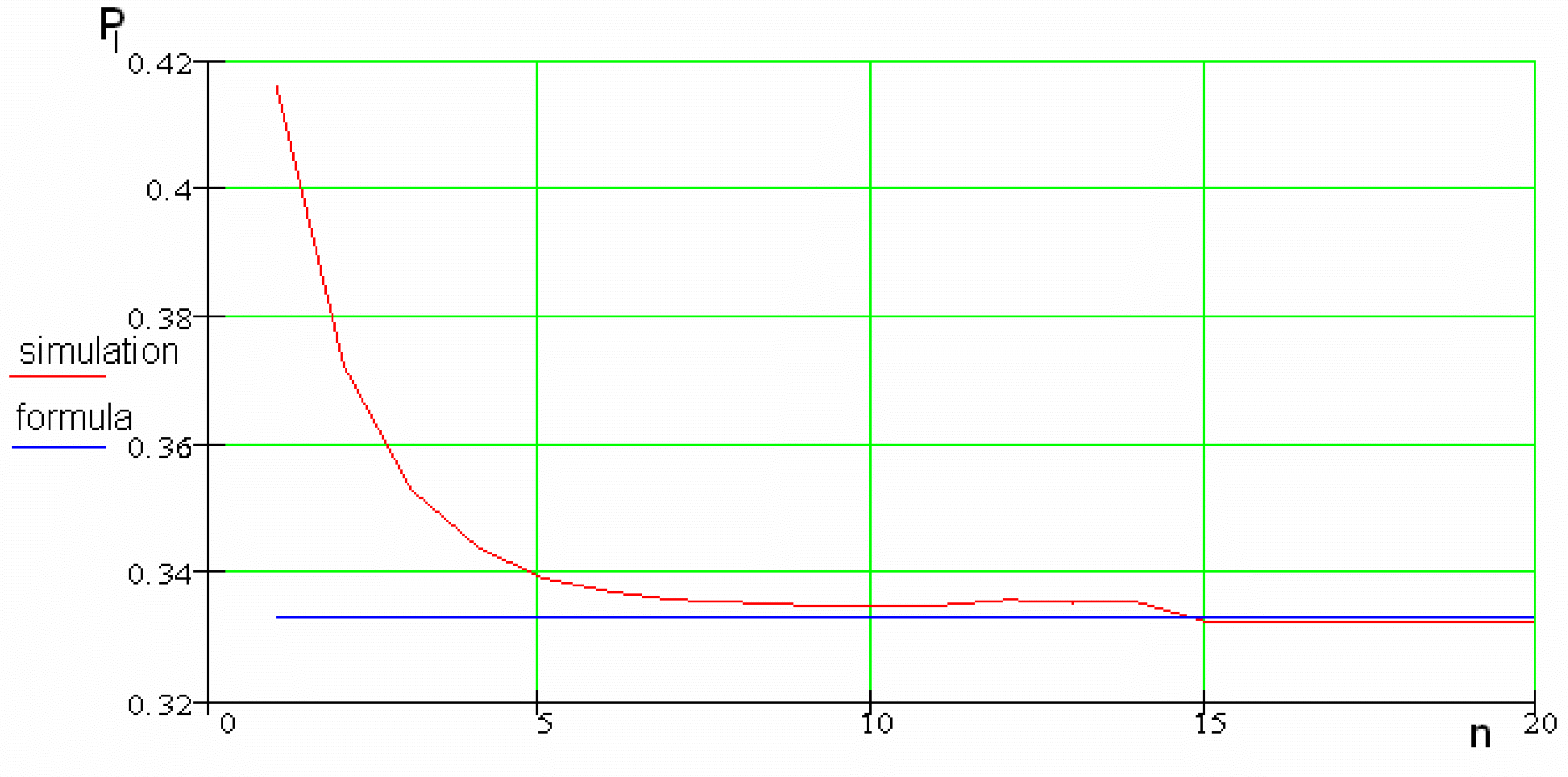
Asymptotic result [Norros]

$$\log P(Q > x) \sim -2 \cdot \frac{C - m}{m} \cdot x$$



Asymptotic result for M/M/1/n [Abramov]

$$P_l \xrightarrow[n\to\infty]{} 1-\frac{1}{\rho},\ \rho>1.$$



Simulation of BM

$$B\left(t_n^h\right) = \Delta_1^h B + \cdots + \Delta_k^h B, \quad k = 1, \ldots, n,$$

where

$$\Delta_n^h B \sim N(0, h) \qquad\qquad h = h_N = 1/N,$$

Error of linear interpolation:

$$\mathbf{E} \int_0^1 \left| B^h(t) - B(t) \right| dt = c / N^{1/2}, \quad c = \sqrt{\pi / 32}$$

Conclusions

- Analysis of queue with Brownian input via regeneration method, based on i.i.d. property of increments of BM.

- Confidence estimation of loss probability.

- Comparison of simulation results with theoretical asymptotic formula.

- Similarity of estimator of P_l for BM/D/1/n and asymptotic results for M/M/1/n.

References

- *Abramov V. M.* One property of a refusals stream// J. Appl. Prob. 1997. Vol. 34, No 3. P. 800-805.

- Asmussen S., Glynn P. Stochastic Simulation: algorithms and analysis. Springer, 2007.

- *Glynn P. W., Iglehart D. L.* Conditions for the applicability of the regenerative method// Management Science. 1993. 39. P. 1108-1111.

- *Law A. M., Kelton W. D.* Simulation modeling and analysis/ New York: McGraw- Hill, 1991. 2nd edt.

- *Mandjes M.* Large Deviations of Gaussian Queues. - Chichester: Wiley, 2007.

- *Norros I.* A storage model with self-similar input, Queueing Systems, vol. 16, pp. 387-396, 1994.

Thank you.