Equilibrium in a P2P-system

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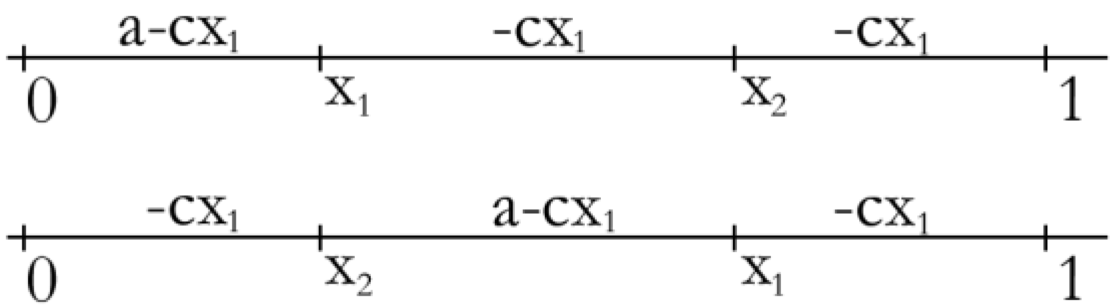
1. Case of two free-riders and one contributor



Player I's payoff if players make contributions x_1 and x_2 and contributor's contribution a is uniformly distributed in $[0,1]$.

$$H_1(x_1,x_2)=\begin{cases}\displaystyle\int_0^{x_1} a da - cx_1 = \frac{x_1^2}{2} - cx_1, & \text{for } x_1 < x_2\\[2ex] \displaystyle\int_{x_2}^{x_1} a da - cx_1 = \frac{x_1^2 - x_2^2}{2} - cx_1, & \text{for } x_1 > x_2\end{cases}$$

Let player II uses the mixed strategy with propability density function $g(x_2)$ in $[b, 1]$ in such a way as to $H_1(x_1, g(x_2)) = v$, where v is a game value. In these conditions player I's payoff is:

$$H_1(x_1, g(x_2)) = \begin{cases} \dfrac{x_1^2}{2} - cx_1, & \text{for } x_1 < b \\ \\ \dfrac{x_1^2}{2} - cx_1 - \dfrac{1}{2}\displaystyle\int_b^{x_1} x_2^2 g(x_2) dx_2, & \text{for } x_1 > b \end{cases}$$

$$\frac{\partial H_1(x_1, g(x_2))}{\partial x_1} = x_1 - c - \frac{x_1^2}{2} g(x_1) = 0.$$

and

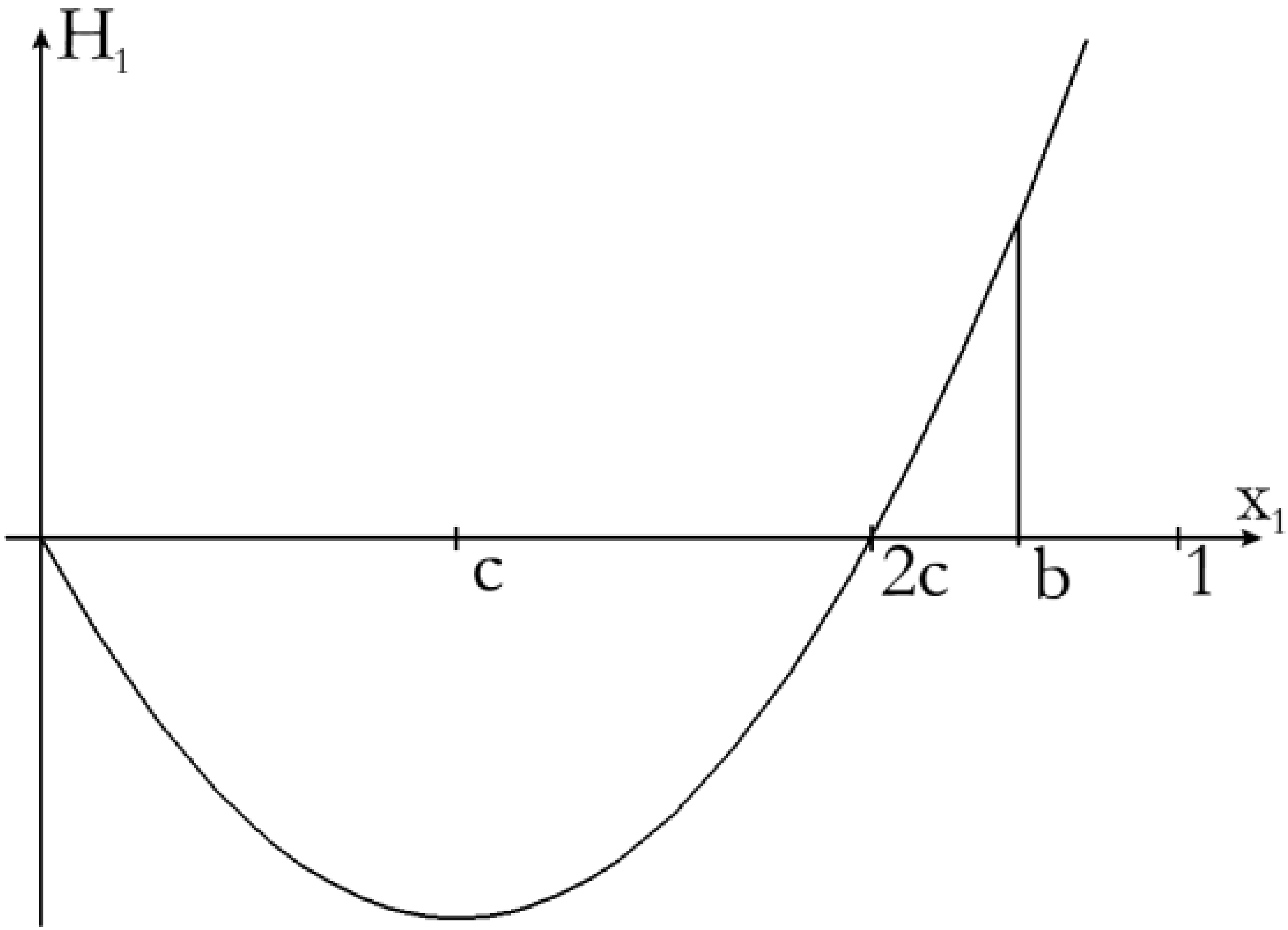
$$g(x_1) = \frac{2}{x_1} - \frac{2c}{x_1^2}.$$

$$G(x) = \int g(x)dx = 2\ln x + \frac{2c}{x}(1-x) + 1 \tag{1.1}$$

$$G(b) = 0 \Rightarrow 2\ln b + \frac{2c}{b}(1-b) + 1 = 0. \tag{1.2}$$

The strategy $G(x)$ is optimal if

$$H_1(x_1, g(x_2)) = x_1^2/2 - cx_1 < b^2/2 - cb \quad \forall x_1 < b. \tag{1.3}$$



Using $b = 2c$ in (1.2) we get condition for c:

$$\ln(2c) + 1 - c = 0 \Rightarrow c^* \approx 0.23196. \qquad (1.4)$$

Since game is symmetric we have prove

Theorem 1: *Let* $c \in [0, c^*]$ *where* c^* *is a root of (1.4). Then the optimal strategies of players are coincide and have form*

$$G(x) = \begin{cases} 0, & \text{for } x < b \\ \\ 2\ln x + \dfrac{2c}{x}(1-x) + 1, & \text{for } x \in [b, 1] \end{cases},$$

where b *satisfies*

$$2\ln b + \frac{2c}{b}(1-b) + 1 = 0$$

$H_i(g, g) = b^2/2 - cb.$

2. Case of two players and n contributors

$$H_1(x_1,x_2)=\begin{cases} n\int_0^{x_1} a da - cx_1 = n\frac{x_1^2}{2} - cx_1, & \text{for } x_1<x_2 \\ n\int_{x_2}^{x_1} a da - cx_1 = n\frac{x_1^2-x_2^2}{2} - cx_1, & \text{for } x_1>x_2 \end{cases}$$

Let player II uses the mixed strategy with propability density function $g(x_2)$ in $[b,1]$ in such a way as to $H_1(x_1,g(x_2))=v$, where v is a game value. In these conditions player I's payoff is:

$$H_1(x_1,g(x_2))=\begin{cases} n\frac{x_1^2}{2} - cx_1, & \text{for } x_1<b \\ n\frac{x_1^2}{2} - cx_1 - n\frac{1}{2}\int_b^{x_1} x_2^2 g(x_2)dx_2, & \text{for } x_1>b \end{cases}$$

$$\frac{\partial H_1(x_1, g(x_2))}{\partial x_1} = nx_1 - c - n\frac{x_1^2}{2}g(x_1) = 0.$$

and

$$g(x_1) = \frac{2}{x_1} - \frac{2c}{nx_1^2}.$$

$$G(x) = \int g(x)dx = 2\ln x + \frac{2c}{nx}(1-x) + 1 \tag{2.1}$$

$$G(b) = 0 \Rightarrow 2\ln b + \frac{2c}{nb}(1-b) + 1 = 0. \tag{2.2}$$

The strategy (2.1) is optimal if

$$H_1(x_1, g(x_2)) = nx_1^2/2 - cx_1 < nb^2/2 - cb \quad \forall x_1 < b. \tag{2.3}$$

Using $b = 2c/n$ in (2.2) we get condition for c:

$$\ln(2c/n) + 1 - c/n = 0. \qquad (2.4)$$

Since game is symmetric we have prove

Theorem 2: *Let $c \in [0, c^*]$ where c^* is a root of (2.4). Then the optimal strategies of players are coincide and have form*

$$G(x) = \begin{cases} 0, & \text{for } x < b \\ 2\ln x + \dfrac{2c(1-x)}{nx} + 1, & \text{for } x \in [b, 1] \end{cases},$$

where b satisfies

$$2\ln b + \frac{2c(1-b)}{nb} + 1 = 0.$$

$H_i(g, g) = nb^2/2 - cb.$

3. Case of m players and n contributors

for $x < b$:

$$H_1(x, g, \ldots, g) = n\frac{x^2}{2} - cx$$

for $x \in [b, 1]$:

$$\begin{aligned} H_1(x, g, \ldots, g) &= -cx + n\frac{x^2}{2}(\overline{G}(x))^{m-1} \\ &+ n\sum_{k=1}^{m-1}\binom{m-1}{k}k(\overline{G}(x))^{m-1-k}\int_b^x g(t)(G(t))^{k-1}\frac{(x^2-t^2)}{2}dt, \end{aligned} \tag{3.1}$$

After some simplification we can rewrite (3.1) as

$$
\begin{aligned}
H_1(x, g, \ldots, g) &= -cx + n\frac{x^2}{2}(\overline{G}(x))^{m-1} \\
&+ n\sum_{k=1}^{m-1} \binom{m-1}{k} (\overline{G}(x))^{m-1-k} \int_b^x t(G(t))^k dt,
\end{aligned} \tag{3.2}
$$

Since players use optimal strategies game value $v = H_1(x, g, \ldots, g)$ for $x \in [b, 1]$. From $\frac{\partial H_1(x,g,\ldots,g)}{\partial x} = 0$ we obtain

for $b < x < 1$ and $\forall m \geq 2$:

$$
\frac{nx - c}{g(x)n(m-1)} = \frac{b^2}{2}(\overline{G}(x))^{m-2} + \int_b^x t(\overline{G}(x) + G(t))^{m-2} dt. \tag{3.3}
$$

Consider the sequence of functions

$$s_k(x) = \frac{2}{x^2}\Big[\frac{b^2}{2}(\overline{G}(x))^k + \int_b^x t(\overline{G}(x)+G(t))^k dt\Big], \quad \forall k = 1, 2, \ldots, m-2, \tag{3.4}$$

which clearly satisfies

$$1 \equiv s_0(x) \geq s_1(x) \geq s_2(x) \geq \ldots \geq s_{m-2}(x) \geq 0, \quad \forall x \in [b, 1].$$

Multiplying $x^2/2$ on the both side of (3.4) and differentiating we get recurrential differential equation

$$\frac{2}{x}(1 - s_k(x)) - s_k'(x) = kg(x)s_{k-1}(x), \quad \forall k = 1, 2, \ldots, m-2 \tag{3.5}$$

with boundary conditions

$$s_k(b) = 0, \quad \forall k = 1, 2, \ldots, m-2.$$

By (3.3)-(3.4) we see

$$s_{m-2}(x) = \frac{2(nx - c)}{nx^2(m-1)g(x)}.$$

From above we obtain $g(x)$

$$g(x) = \frac{2(nx-c)}{nx^2(m-1)s_{m-2}(x)} \geq \frac{2(nx-c)}{nx^2(m-1)}.$$

We get b from

$$\int_b^1 g(x)dx = 1. \tag{3.6}$$

For optimality it is nesessary that $nx_1^2/2 - cx_1 < nb^2/2 - cb \quad \forall x_1 < b$ or equivalent $b > 2c/n$. Using $b = 2c/n$ in (3.6) we get condition for c:

$$\int_{2c/n}^1 g(x)dx = 1. \tag{3.7}$$

Hence we have prove

Theorem 3: *Let* $c \in [0, c^*]$ *where* c^* *is a root of*

$$\int_{2c/n}^{1} g(x)dx = 1.$$

Also let $\{s_1, \ldots, s_{m-2}\}$ *is a solution of the system of differential equations*

$$\frac{2}{x}(1 - s_k(x)) - s_k'(x) = kg(x)s_{k-1}(x), \quad \forall k = 1, 2, \ldots, m-2$$

with boundary conditions

$$s_k(b) = 0, \quad \forall k = 1, 2, \ldots, m-2.$$

and

$$g(x) = \frac{2(nx - c)}{nx^2(m-1)s_{m-2}(x)}.$$

Let us choose b *from condition* $\int_b^1 g(x)dx = 1$. *Then* $g(x)$ *is an optimal strategy.*

$$H_i(g, g, \ldots, g) = nb^2/2 - cb.$$