Verification of the long-range dependent property in the tandem networks

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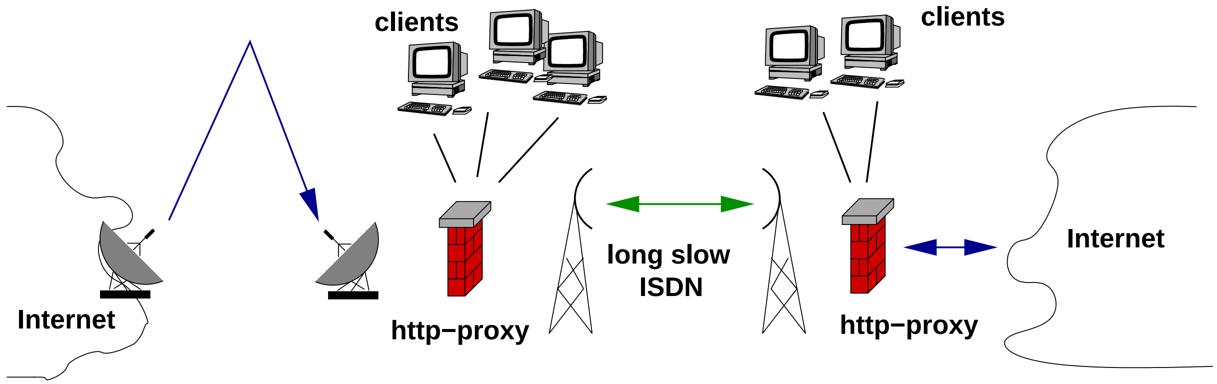
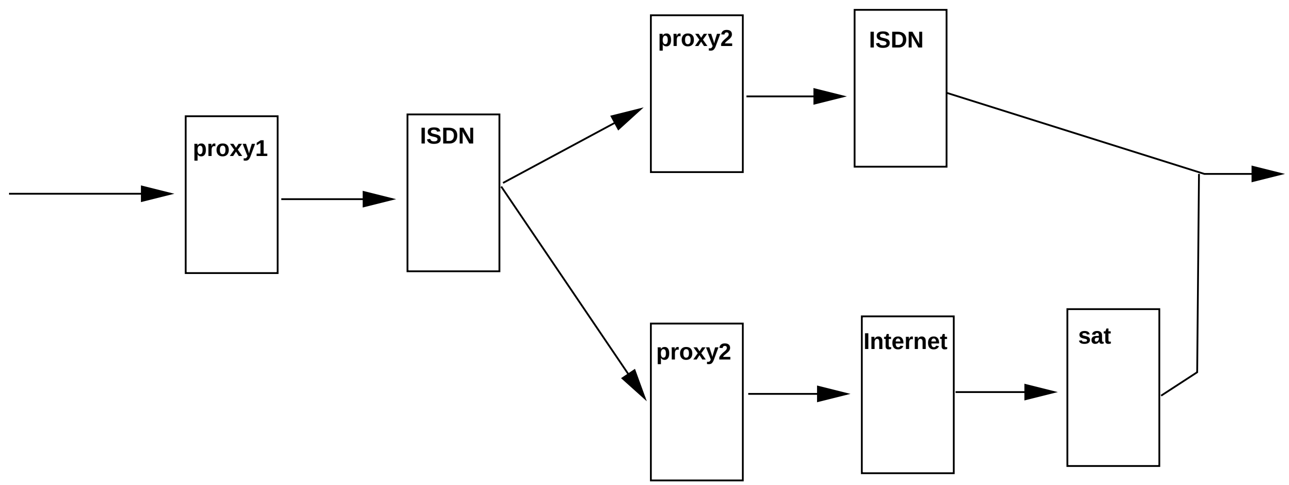




Model of the 'real'-network.

In this work we will consider tandem networks $G/G/1 \to \cdots \to \cdot/G/1$ and branching networks. Tandem contains two or more single-channel stations and every customer goes through the stations by some pre-defined route.

The simple single station queue $G/G/1$ is a well-studied in the literature. Our main goal in this work is to verify by simulation Long-Range Dependence of the workload proccess at the all stations of the tandem network under the same conditions which imply the Long-Range Dependence of the workload proccess in the simple single station queue. We expect to detect the similar effect on all station of the tandem network.

Also we show that it is possible to apply regenerative approach for estimation of the queuing characteristics in such systems.

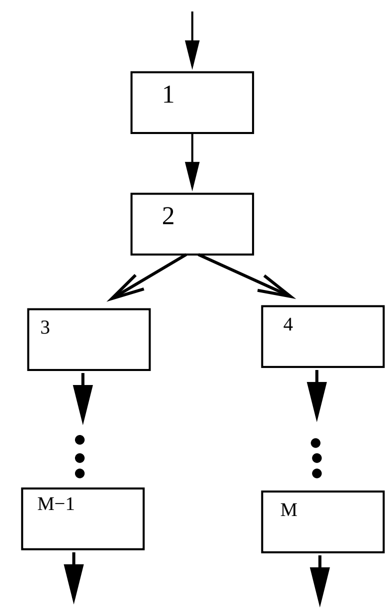


Figure: Multi-station Tandem.

Simple $G/G/1$ queue

Let the queue $G/G/1$ has renewal input $\{\tau_n = t_{n+1} - t_n\}$, where t_n is the arrival instant of customer n, and i.i.d. service times $\{S_n\}$, and $\rho = ES_1/E\tau_1 < 1$. W_n is waiting time of customer $n = 1, 2, \ldots$, T_n is a departure time.

D. Daley (1969) studied a stationary single server queue $G/G/1$. It has been proved that under following moment conditions for S and τ:

$$ES_1^3 < \infty,\ ES_1^4 = \infty \ \text{ and } \ E\tau_1 < \infty,$$

the workload process of the queue $\{W_n\}$ is **Long-Range Dependent**:

$$\sum_{n=0}^{\infty} \operatorname{cov}(W_0, W_n) = \infty.$$

A stationary sequence $\{D_n\}$ of inter-departure intervals is defined by

$$D_n = T_{n+1} - T_n = \tau_n + W_{n+1} + S_{n+1} - W_n - S_n$$

It has been proved by D. Daley and R. Vesilo (1997) that if $ES^2 < \infty$ and $E\tau^2 < \infty$, then

$$\lim_{n\to\infty} \sum_{j=1}^{n} \operatorname{cov}(D_0, D_n) = \frac{1}{2}\left[\mathbf{var}\tau - \mathbf{var}D\right].$$

Tandem network

Tandem networks $G/G/1 \to \ldots \to \cdot/G/1$ consists of several single-servers stations and every customer goes through the network by some route.

We denote $t_n^{(0)} = t_n$ that is the input instants of the arrival n for the first station. Waiting times and arrival and departure moments for each server (where M is the total number of servers) can be computed recurrently as follows
$(W_0^{(j)} = 0, \quad j = 2, 3 \ldots, M)$

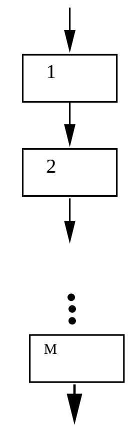
$$t_n^{(j)} = t_n^{(j-1)} + W_n^{(j-1)} + S_n^{(j-1)},$$

$$W_{n+1}^{(j)} = (W_n^{(j)} + S_n^{(j)} - (t_{(n+1)}^{(j)} - t_n^{(j)}))^+$$

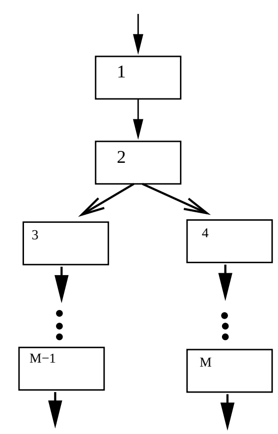
The departure time from the network:

$$T_n = t_n^{(M)} + W_n^{(M)} + S_n^{(M)}. \qquad n \geq 1.$$

(Indecies reordering is requeired for the branching networks.)



Simple multi-station tandem.

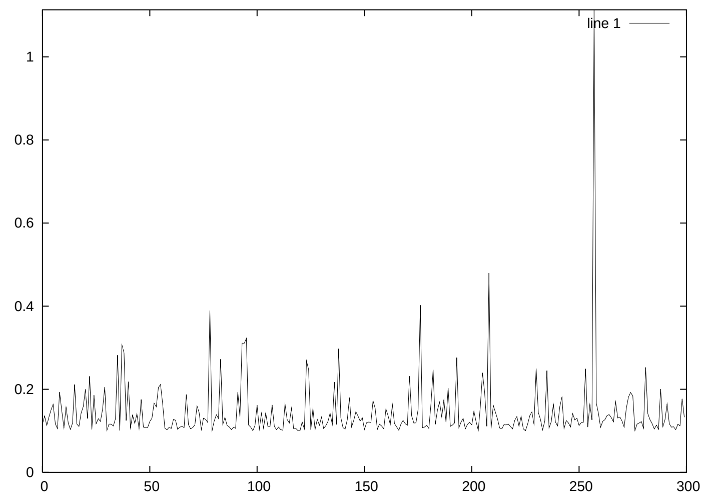


Branching tandem network.

Simulation

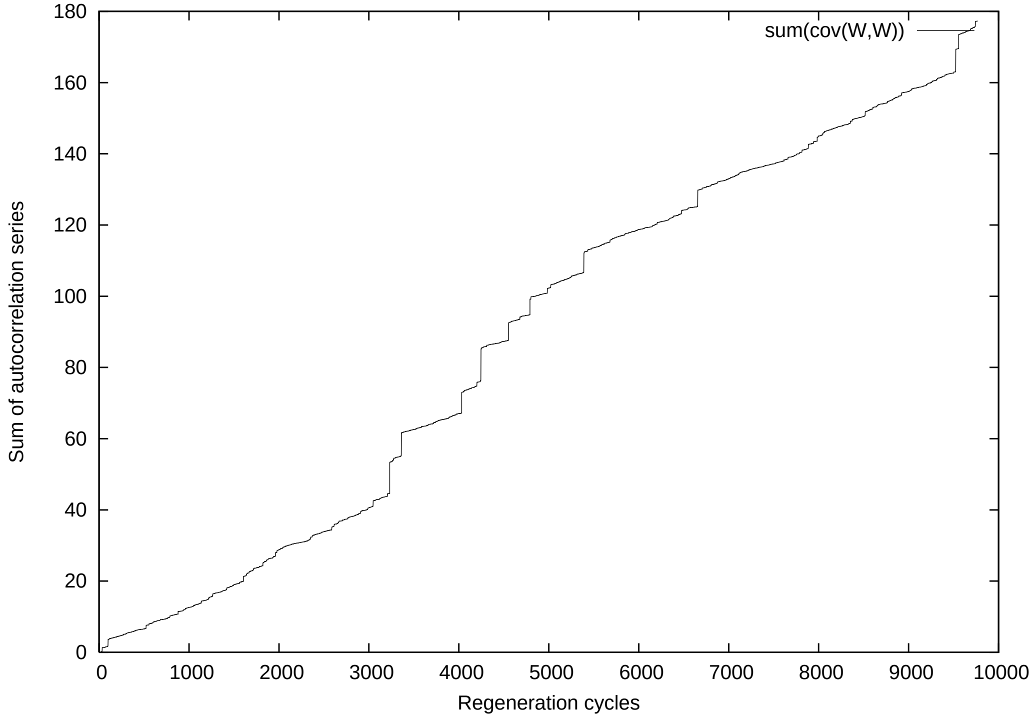
Simulation:

- Poisson disribution was used to simulate the sequence of arrival intervals
- Pareto service times was used for all servers with parameters $\gamma_1 = \ldots = \gamma_M \in (3, 4)$, to satisfy condition $ES^3 < \infty, ES^4 = \infty$.
- $\rho = \frac{ES}{E\tau} \geq 0.8$



Example of Pareto service time realizations, $\gamma = 3.5$

Autocorrelation series



Autocorrelation series $\sum \mathrm{cor}(W_0, W_n)$, $M = 4$, for the station 4 in the non-branching tandem network.

It is more convenient from the computation point of view to compute the estimate of autocorrelation coefficients (Chatfiel, 1978):

$$\mathrm{cor}(W_0, W_n) = \frac{\frac{1}{N-n}\sum\limits_{t=1}^{N-n}(W_t - \frac{1}{N}\sum\limits_{i=1}^{N} W_i)(W_{t+n} - \frac{1}{N}\sum\limits_{i=1}^{N} W_i)}{\frac{1}{N}\sum\limits_{t=1}^{N}(W_t - \frac{1}{N}\sum\limits_{i=1}^{N} W_i)^2}.$$

Since the $\mathrm{var}\tau < \infty, \mathrm{var}S < \infty$ autocovariance and autocorrelation series converge or diverge simultaneously.

Regeneration instantces

Regeneration points $\{\beta_n\}$ are defined as such instant t_n when arrival n meets an empty queue, that is $W_n = 0$, or in other words.

$$\{\beta_k\} = \{n : t_n^{(0)} > T_{n-1}\}$$

It can be proved that if $ES_1^3 < \infty$, $E\tau^3 < \infty$ and $\rho < 1$ then busy period $\alpha_n = \beta_{n+1} - \beta_n$ has finite 3rd moment. And the stationary forward regeneration time $\alpha(t)$ (at arbitrary instant t) has a finite second moment, and $P(\alpha(t) > x) = o(x^{-2}) \quad \text{as} \quad x \to \infty$. This indicates that it is possible to accumulate a necessary number of regeneration points in an acceptable simulation time.

Denote $\Delta_n = \beta_{n+1} - \beta_n$; $\bar{\Delta}_n = \frac{1}{n}\sum \Delta_i$; $r_n = \frac{\bar{Y_n}}{\bar{\Delta}_n}$, where $\bar{Y_n} = \frac{1}{n}\sum Y_i$, and Y_j is the sample mean of **the total waiting time** in the queue for all customers at the j-th cycle:

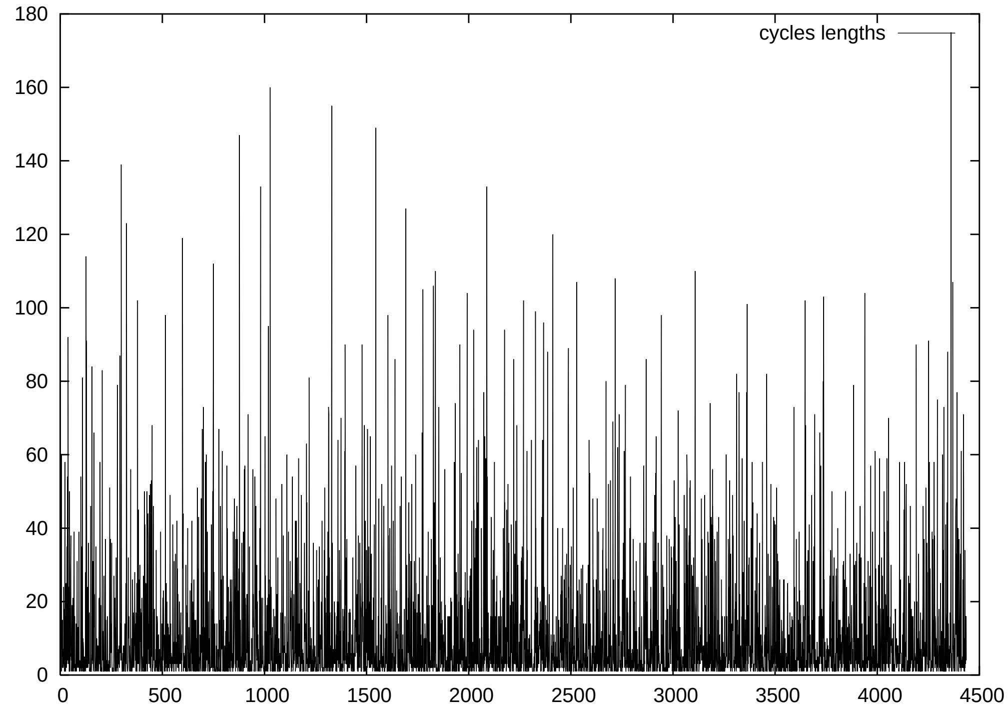
$$Y_j = \sum_{i=\beta_j}^{\beta_{j+1}-1} (W_i^{(1)} + \ldots + W_i^{(M)}) \qquad j \geq 0.$$

Then $100(1-\delta)\%$ confidence interval in the case of classical regeneration can be written as

$$I = \left[\hat{r} - \frac{z_\delta S(n)}{\sqrt{n}\bar{\Delta}_n}; \hat{r} + \frac{z_\delta S(n)}{\sqrt{n}\bar{\Delta}_n}\right],$$

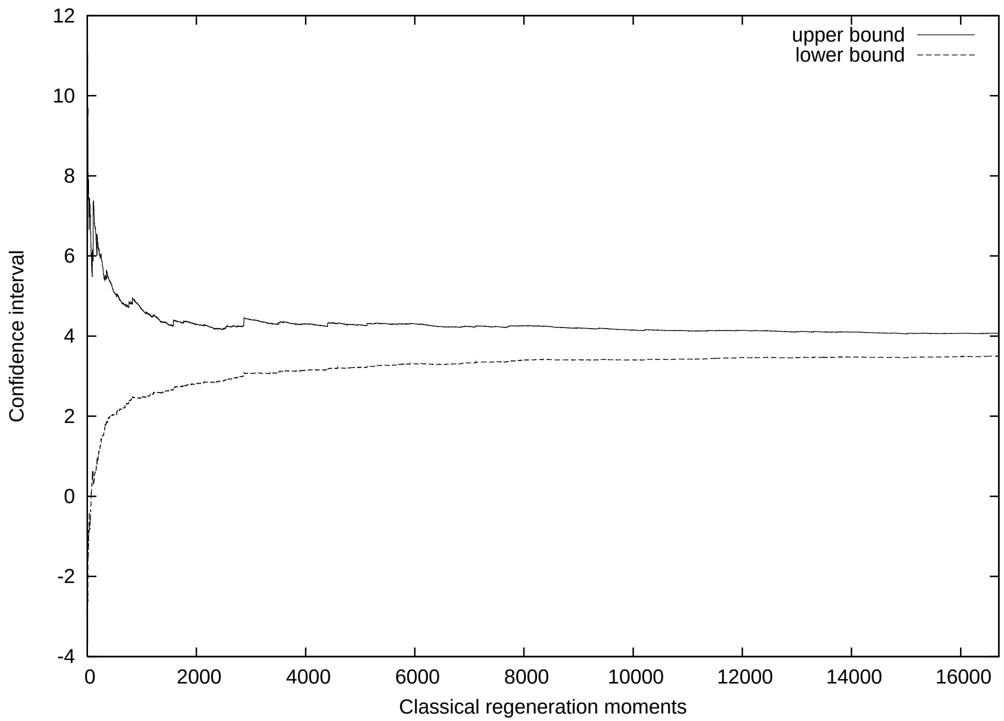
where z_δ is the δ-quantile of the standard normal variable $N(0,1)$. Quantity $S^2(n)$ is the standard estimator of the variance $\sigma^2 = E(Y_1 - r\Delta_1)^2$.

Regeneration cycles



Length of the regeneration cycles in thenon-branching tandem network, $M = 4$.

Confidence interval



Confidence interval for the average total wworkload $E(W^{(1)} + \cdots + W^{(M)})$; $M = 4$, $z_\delta = 0.95$.

Quasi regeneration

Quasi-regeneration moments generated by the customers crossing the tandem network without collisions.
Algorithmically for the non-branching network they may be defined as such instants

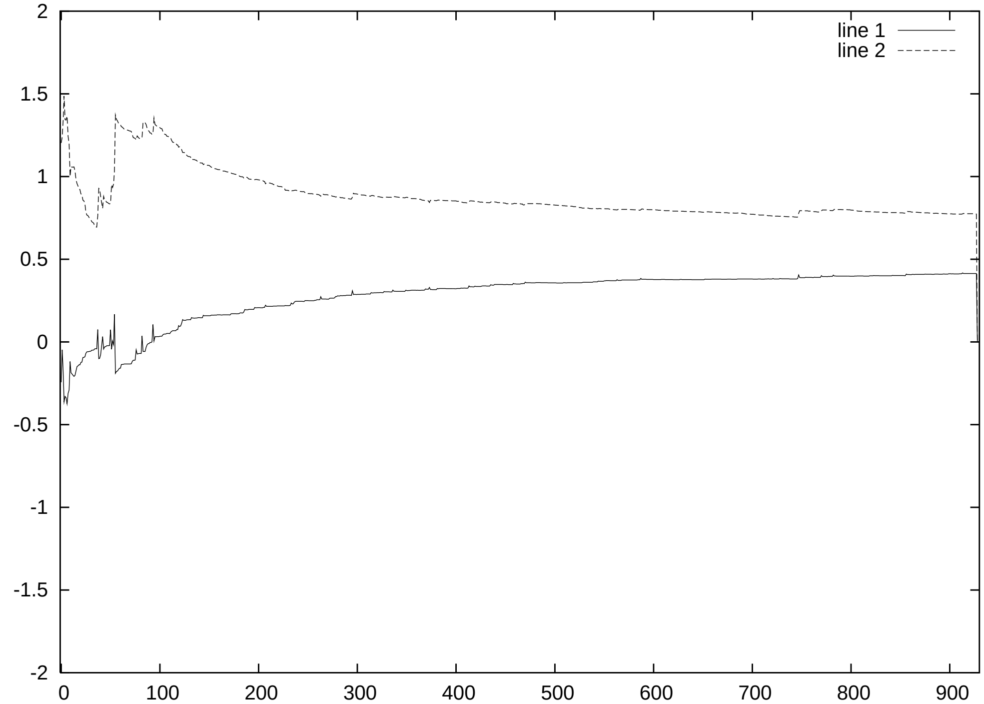
$$\{\beta_k\} = \{n : t_n^{(j-1)} > t_{n-1}^{(j)}, \quad j = 1, \ldots, M, i = 1 \ldots N\}$$

Then confidence interval for the quasi-regeneration can be written as

$$I = \left[\hat{r} \pm \frac{z_\gamma \sqrt{S^2(n) + 2(t_1(n) - r_n t_2(n))}}{\sqrt{n}\bar{\Delta}_n}\right],$$

where $t_1(n)$, $t_2(n)$ are standard estimators of $\mathrm{cov}(Y_1,\, Y_2)$ and $\mathrm{cov}(\Delta_2,\, Y_1)$, respectively.

Confidence interval



Confidence interval for the average total workload for the quasi-regeneration $E(W^{(1)} + \cdots + W^{(M)})$; $M = 3$, $z_\delta = 0.95$.

Conclusion

- Our simulation results confirm that the stationary workload process W_n exhibits long-range dependence at every station in the tandem network under the same condition as for the simple $GI/G/1$ queue.
- We had build the confidence interval for the total workload time with the regenerative approach.