The Decomposition Property of the Blocking Queueing Model in a Random Environment

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Abstract

Several recent results on the queueing systems involving the random environment state that some models of this class allow decomposition. For these cases, steady state joint distributions of certain interest might be presented as a product of probabilities characterizing separately the state of random environment and the state of the queueing system. In this paper we consider a generalized heterogeneous blocking queueing scheme in the steady state. For the scheme we obtain necessary and sufficient conditions which guarantee the correctness of the decomposition. We also obtain the joint distribution of the state of the queueing system and the state of the random environment in the analytical product form and prove that the obtained distribution is independent of the service times distribution.

1 Introduction

Many areas appeal to stochastic models whose essential parameters vary randomly over time, depending on the state of some external stochastic process. The external process is usually named the random environment.

In the telecommunication area these models undoubtedly are of great interest, as they allow a series of practically important interpretations. The most obvious examples of the case are systems whose packets or connections arrival rate, service rate or service scheduling may change during the observation period. These models also allow to investigate the cases of server breakdown or the rush hour phenomena. Certainly the set of possible interpretations is much wider than the given examples describe.

Usually the queues whose parameters vary randomly over time are named queues involving in a random environment. Typically these models possess a high complexity. Therefore the corresponding analysis raise significant challenges and results are often cumbersome and insufficient.

Nevertheless, there is a class of blocking queuing systems in a random environment which allows obtaining explicit analytical solutions. In the cases one may acquire the exact product forms for the system's important characteristics like joint distribution of the state of a random environment and the busy servers number. At the same time, the comparatively simple analytical results are obtainable only under special restrictions on the parameters of arrival and departure processes.

Another important fact is that some of the models mentioned above possess the invariance property (see for instance [3]). This is also a significant direction of the analysis since the invariance property widens the usability of the model.

The model considered in this paper might be treated as a generalization of the queuing blocking system investigated in [3] and [7]. We analyze a general heterogeneous blocking system in a random environment with state dependent arrival. We have formulated the necessary and sufficient conditions which allow presenting the joint steady state distribution of the external environment state and the state of the system in an explicit product form. The state of the system is formulated as a vector which reflects the nature of the system's heterogeneity. We use a random ergodic Markovian process with discrete space of states as a model of the external environment. The ergodic property of the process allows us to consider the steady state characteristics of the whole model.

The rest of the paper is organized as follows. Section 2 contains the description of the queuing system under investigation and the formalization of the problems considered. The necessary and sufficient conditions which provide the existence of the product forms for the distribution introduced in section 2 are formulated in theorem in section 3. The proof of the theorem also shows that the theorem conditions provide the invariance property for the model and the distribution obtained in the product form does not depend on the service times distribution. Section 4 includes some remarks and discussion. Using simple random time change, we present an extended detailed analysis and interpretation of the established conditions.

2 Description of the Model

The considered model consists of two parts. The first is an external environment and the second is the same service system. The external environment is described as an ergodic continuous time Markovian process $\xi(t)$ with a discrete state of space Y and the rates of the transitions $q_{yy'}\colon y, y` \in Y$. The steady state distribution of $\xi(t)$ is set as a sequence $\{\pi_y\}_{y\in Y}$.

The queue consists of several groups of the servers. All servers in a group are of the same service ability. Let us define the state of the system as vector $N,\ \ n_i \in N,\ i = 1\dots s$ where s is the general number of device groups and n_i is the number of busy servers in a group i. If the system is found in the state N at a time moment t, then the probability of new customer arrival inside the term $t + \Delta t$ in the group i is defined as $\lambda_i(y, N)\Delta t + o(\Delta t)$, where $y \in Y$. The probability of two or more customers arriving inside the term Δt has the order $o(\Delta t)$. Let us denote the number of servers in the group as m_i. If the customer arrives in the service group, it chooses a free server arbitrarily.

The customer service times are realizations of the random variables η_i and depend on the number of group which serves the customer and the state of the external environment. The service rates are correspondingly defined as $\mu_i(y)$. We do not specify the service times distribution function, so any function may be assumed.

If the customer arriving in the service group finds all its servers busy, it is lost. Let us name the described queue Σ_{env} (Figure 1).

Let us also associate with the state of the system the random process $\eta(t) = \{\xi(t), N(t)\}_{t>0}$, where $N(t)$ is the state of the service system at a time t. Let Q denote the set of vectors N that the states of the system

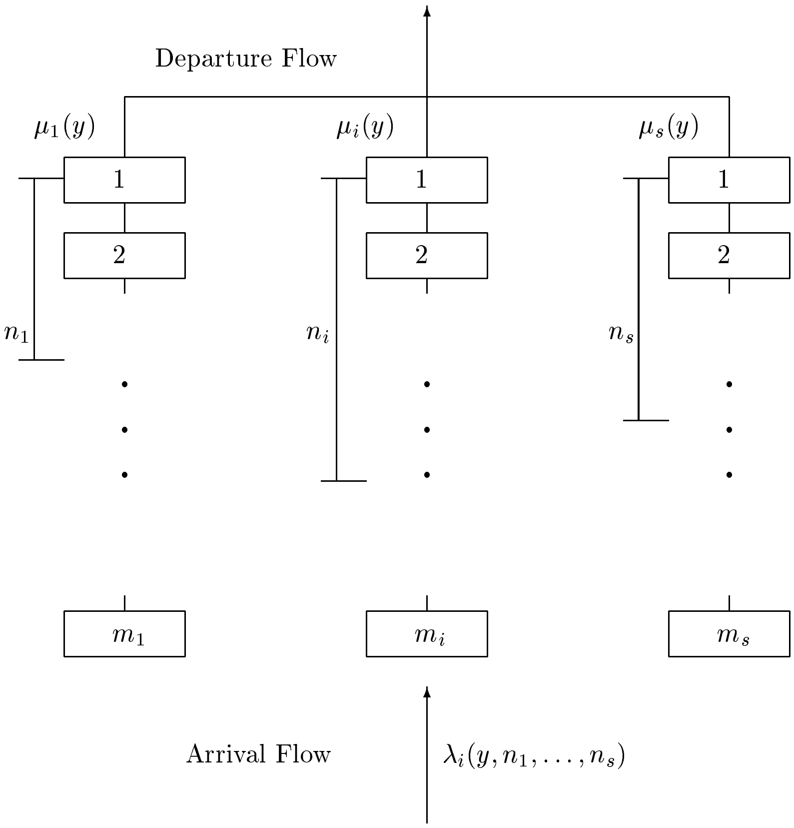

Figure 1. *Queueing system* $\Sigma_{\rm env}$

allow. The space of states of the process $\eta(t)$ is a union of set Y and set Q. The steady state distribution of the process $\eta(t)$ always exists due to the general theorems formulated by A.Borovkov [1]. We will consider the following limits

$$p(y, N) = \lim_{t \to \infty} Pr(\xi(t) = y \ , \quad N(t) = N) \ .$$

The limits define the steady state distribution of $\eta(t)$.

3 The main result

Assume that there is a function $g(y)$ such that

$$\lambda_i(y,N) = g(y)v_i(N) \tag{3.1}$$

and a set of constants $\{\tau_i\}_{i=1}^s$ such that

$$\mu_i(y) = g(y)\tau_i \ . \tag{3.2}$$

Also assume that $v_i(N)$ satisfies equality

$$\prod_{l=1}^{n_1+\cdots+n_s} v_{i_l}\left(\sum_{m=1}^{l-1}\delta_{1,i_m},\ldots,\sum_{m=1}^{l-1}\delta_{s,i_m}\right) = \\ = \prod_{l=1}^{n_1+\cdots+n_s} v_{j'_l}\left(\sum_{m=1}^{l-1}\delta_{1,i'_m},\ldots,\sum_{m=1}^{l-1}\delta_{s,i'_m}\right) , \tag{3.3}$$

where $\{n_1,\ldots,n_s\}$ is any available vector $N \in Q$; $\{i_1,\ldots,i_{n_1+\cdots+n_s}\}$ and $\{i'_1,\ldots,i'_{n_1+\cdots+n_s}\}$ are any two set and $\delta_{ij}=1$ if $i=j$ or $\delta_{ij}=0$ if $i\neq j$.

Formula (3.3) means that in average customers do not prefer a particular group of servers. For all vector $R \in Q$ so that $r_i \leq n_i$, $i = 1,\ldots,j-1,j+1,\ldots,s$, and $r_j+1 \leq n_j$, this condition may be reduced to the following one:

$$v_i(N)v_j(N+e_i) = v_j(N)v_i(N+e_j) \ ,$$

i.e. any way of accumulation of the product V(N) gives the equal result or the results differ smaller than $o(\Delta t)$.

Theorem 1. *The random process $\eta(t)$ has the steady state distribution of the following form*

$$p(y,N) = p_0\pi_y\frac{V(N)}{\prod\limits_{i=1}^{s} n_i!\tau_i^{n_i}} \ , \tag{3.4}$$

if and only if $\lambda_i(y,N)$ and $\mu_i(y)$ satisfy (3.1), (3.2) and (3.3). Here p_0 is a normalizing constant and $V(N)$ is the left (right) part of (3.3).

Proof. *Necessity.* Let us consider the case of exponentially distributed service times. The distributions parameter is $\mu_i(y)$[1]. The case $\eta(t)$ is a regular Markovian process. If (3.4) is the steady state distribution of $\eta(t)$ it must satisfy the global balance equations (i.e. Kolmogorov equations). The process may reach state (y, N) only from the states:

- ▹ $(y, N - e_i)$ with the rate $\lambda_i(y, N)$;
- ▹ $(y, N + e_i)$ with the rate $\mu_i(y)(n_i + 1)$;
- ▹ (y', N) with the rate $q_{y'y}$.

Hence the global balance equations system is

$$-p(y,0)\sum_{i=1}^{s}\lambda_i(y,0) + \sum_{i=1}^{s}\mu_i(y)p(y,0+e_i) + \sum_{y'\in Y} q_{y'y}p(y',0) = 0 \quad (3.5)$$

$$\sum_{i=1}^{s}\Big[\lambda_i(y,N-e_i)p(y,N-e_i) - (\lambda_i(y,N)I(n_i) + n_i\mu_i(y))p(y,N) + \\ + (n_i+1)\mu_i(y)p(y,N+e_i)\Big] + \sum_{y'\in Y} q_{y'y}p(y',N) = 0 \ , \quad (3.6)$$

where $I(n_i)$ is 0 if $n_i = m_i$ and 1 otherwise. Equations (3.5) describe the system's behavior if it is idle. Equations (3.6) are valid for all the rest of the states.

Let us also examine the partial balance equations [5]. The following one allows deriving conditions (3.1) and (3.2)

$$p(y,N)\sum_{y\neq y'} q_{yy'} = \sum_{y'\neq y} p(y',N)q_{y'y} \ .$$

The last formula presents the Kolmogorov equation for the process $\xi(t)$, which has a unique solution up to the constant multiplier. The solution is the steady state distribution π_y. Hence

$$p(y,N) = \pi_y P(N) \ ,$$

[1] Actually in general case, the service times distribution is the convolution of exponential distributions with parameters $\mu_i(y)$, $\mu_i(y')$, $\mu_i(y'')$ etc., since the state of the environment may change during the customer service time.

where $P(N)$ is expression independent on y. To satisfy the global balance equations this expression must be a product whose factors are of the following form

$$\frac{\lambda_i(y,N)}{\mu_i(y)}$$

in correspondence with N. If products $P(N)$ for all N are independent of y then any of those factors must be independent of y and hence conditions (3.1) and (3.2) take place.

Now let us consider another set of partial balance equations

$$p(y,n)\lambda_i(y,N) = p(y,N+e_i)(n_i+1)\mu_i(y)\ , \quad i = 1,\ldots,s\ .$$

The substitution of (3.4) yields

$$\frac{V(N)}{\prod\limits_{i=1}^{s} n_i!\tau_i^{n_i}} v_i(N) = \frac{V(N+e_i)}{(n_i+1)\tau_i \prod\limits_{i=1}^{s} n_i!\tau_i^{n_i}} (n_i+1)\tau_i\ . \tag{3.7}$$

Since the denominators of the left and right part of (3.7) are equal

$$V(N)v_i(N) = V(N+e_i)\ . \tag{3.8}$$

The last equality must be true for any cases of i and N and hence (3.3) takes place.

The last set of partial balance equations

$$p(y,N)n_i\mu_i(y) = p(y,N-e_i)\lambda_i(y,N)\ , \quad i = 1,\ldots,s$$

also yields an expression equivalent to (3.8)

$$V(N) = V(N-e_i)v_i(N)\ .$$

Sufficiency. Now we will investigate the case of the general service times distribution to prove that the considered system possesses an invariance property. Our reasoning will be based on the general theory of insensitivity of stationary characteristics for a generalized semi-Markov process [2, 6]. The stochastic processes of this type are constructed by extending the starting state of space in order to use the techniques of Markov processes to study the initial object.

According to the terminology used in [2] the active events associated with the state (y,N) of the process $\eta(t)$ are the following

1. Services of customers at busy servers. The processing rate is $\mu_i(y)$.
2. Expectation of new customer arrival. The processing rate is $\lambda_i(N)$.
3. Expectation of the transition of the $\xi(t)$ random process. The processing rate is $q_{yy'}$.

According to our starting assumptions the service of customers at busy servers are non-exponential active events among those enumerated above. Hence one may associate the following labels to the transitions of the random process $\eta(t)$. The system may reach the following points from the point (y, N):

- ▹ $(y, N + e_i)$ with the rate $\lambda_i(y, N)$. The label of the corresponding transition is N;
- ▹ $(y, N - e_i)$ with the rate $n_i\mu_i(y)$. The label is $N - e_i$;
- ▹ (y', N) with the rate $q_{yy'}$. The label is N.

The transitions which allow reaching point (y, N) were described above. The corresponding label is N for the transitions $(y, N + e_i) \to (y, N)$ and $(y', N) \to (y, N)$. The transition $(y, N - e_i) \to (y, N)$ has the label $(N - e_i)$.

Now according to the main result of [2] we formulate the restricted flow equations (RFE). The RFE means that the flow moving into the point through the transitions with some fixed label is equal to the flow moving from the point through the transitions with the same label. For the considered queueing scheme the RFE are the following[2]

$$p(y,n)\left[\sum_{i=1}^{s}\lambda_i(y,N)+\sum_{y\neq y'}q_{yy'}\right]=$$
$$=\sum_{i=1}^{s}p(y,N+e_i)(n_i+1)\mu_i(y)+\sum_{y'\neq y}p(y',N)q_{y'y} \quad (3.9)$$

$$p(y,N)n_i\mu_i(y)=p(y,N-e_i)\lambda_i(y,N)\ , \quad i=1,\dots,s\ . \quad (3.10)$$

[2]D. Y. Burman [2] notes that restricted flow equations in many contexts are the same as the partial balance equations.

The substitution of (3.4) as $p(y,N)$ in (3.9)–(3.10) turns RFE to equality if the parameters of the system satisfy conditions (3.1), (3.2) and (3.3). This means that under the theorem conditions global balance equations and restricted flow equations have the identical solution and hence according to [2] we can guarantee the invariance property for steady state distribution of $\eta(t)$. □

4 Discussion

In the previous section we have illustrated what the condition (3.3) means. Now we are going to discuss the sense of the conditions (3.1) and (3.2). These conditions allow the above-mentioned decomposition of the steady state distribution

$$p(y,N) = \pi_y P(N) \ . \tag{4.1}$$

Let us consider random time change

$$t^* = \int_0^t g(\xi(\phi))d\phi \ .$$

According to (3.2) for the new time t^* the service will be done with the constant rate. After the time change the service rate depends only on the number of server groups and the mean service time is equal to τ_i. The arrival process parameters are also to be transformed. For the time t^* according to (3.1)

$$\lambda_i^*(y,N) = \frac{\lambda_i(y,N)}{g(y)} = v_i(N) \ .$$

Now it is evident that under conditions (3.1) and (3.2) the state of the system processes independently over time t^*. Note that considered assumptions mean that the "utilization factor"[3] (in general sense) of the

[3]Traditionally the utilization is treated (for instance for $G/G/1$ queue) as a product of average arrival rate times the average service time each customer requires. In the considered case these two parameters vary randomly and therefore we have quoted the term. We mean here a function of "utilization" which depends on i and N.

queueing system also does not depend on the state of the random environment. A possible actual example of this case is a system which reacts to the external factors. Such a system increases the service ability if the arrival flow rate increases or decreases so that if arrival flow weakens it will keep some of the performance characteristics at the given level. We have proved that the decomposition (4.1) holds if and only if the function $g(y)$ exists and satisfies the theorem conditions. The condition (3.3) does not affect the decomposition correctness but it is necessary to hold the invariance property. A full combination of these conditions provides the existence of an explicit analytical solution.

Assume that parameters $g(y) \equiv \text{const}$. This case reduces the considered problem to the general blocking scheme investigated by I. Kovalenko [4, chapter 5] and so does the condition (3.3). In the paper [8] the reduced scheme is also considered with regards to the internal structure of the customers. The special case of the function $\lambda_i(y, N)$ (i.e. $\lambda_i = \lambda(y, \sum_{i=1}^{s} n_i)$) was analyzed in [3]. For the case conditions (3.1) and (3.2) one can transformed formula (3.4) to the form given in [3].

5 Conclusion

In this paper we have studied the class of queueing blocking models involving the random environment. The queueing scheme under investigation consists of two parts. These are the external stochastic process (environment) and the service system. The random environment is treated as an ergodic discontinuous Markovian process. The service system is a heterogeneous multichannel queue with loss and state dependent arrival. The analysis was done in two directions. The first is to establish the necessary and sufficient conditions under which the model allows decomposition. The second is to investigate the invariance property of the model.

With these aims we have formulated and proved the theorem on the existence of an analytical product form for the joint steady state distribution of the state of the queueing system and the state of the random environment. The distribution is obtained in the analytical form. To prove the invariance property we have based the argument on the general theory of insensitivity for the generalized semi-Markov processes.

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